MAT 371: Homework 4 (02/05)

1 Chapter 2.2

Ex 7. [3pts]

a) The sequence $a_n = (-1)^n$ is *frequently* in the set 1. For any N, we have a_n or a_{n+1} in 1. [1pt]

b) *Eventually* is strongly than *frequently*. If $a_n \in A$ for all $n \leq N$, then it is also *frequently* in A. [1pt]

c) We have that $a_n \to a$ if for any neighborhood $V_\varepsilon(a)$ the sequence (a_n) will be *eventually* in $V_\varepsilon(a)$.

d) The sequence (x_n) is *frequently* in the interval $(1.9, 2.1)$ but not *necessarily* for instance $x_n = (-2)^n$. [1pt]

2 Chapter 2.3

Ex 2.

We suppose that $x_n \to 2$.

a) Let $a_n = \frac{2x_n - 1}{3}$, we want to show that $a_n \to 1$. Fix $\varepsilon > 0$. We know that there exists N such that $|x_n - 2| < \varepsilon$ for $n \geq N$. We deduce:

$$|a_n - 1| = \left|\frac{2x_n - 4}{3}\right| = \frac{2}{3}|x_n - 2| < \frac{2}{3}\varepsilon < \varepsilon.$$

Thus, we can use the same N to deduce that $|a_n - 1| < \varepsilon$ if $n \geq N$.

b) Let $a_n = 1/x_n$. Since $x_n \to 2$, there exists N_1 such that $|x_n - 2| < 1$ (take $\varepsilon = 1$), in particular $|x_n| > 1$ and therefore

$$\left|\frac{1}{x_n}\right| < 1 \quad \text{for } n \geq N_1.$$

Fix $\varepsilon > 0$. Take N_2 such that $|x_n - 2| < \varepsilon/2$ for $n \geq N_2$. Consider now $N = \max(N_1, N_2)$. For $n \geq N$, we have:

$$\left|a_n - \frac{1}{2}\right| = \left|\frac{2 - x_n}{2 \cdot x_n}\right| < 2 \cdot |2 - x_n|$$

since $n \geq N_1$. We deduce:

$$\left|a_n - \frac{1}{2}\right| < 2 \cdot |2 - x_n| < \varepsilon.$$

Ex 3.

Since $x_n \leq y_n \leq z_n$, we have:

$$x_n - l \leq y_n - l \leq z_n - l$$

thus $|y_n - l| \leq \max(|x_n - l|, |z_n - l|)$.

Fix $\varepsilon > 0$. Since $x_n \to l$, there exists N_1 such that $|x_n - l| < \varepsilon$ for $n \geq N_1$. Similarly there exists N_2 such that $|z_n - l| < \varepsilon$ for $n \geq N_2$. Take now $N = \max(N_1, N_2)$. For any $n \geq N$, we have:

$$|y_n - l| \max(|x_n - l|, |z_n - l|) < \max(\varepsilon, \varepsilon) = \varepsilon.$$

Ex 5. [3pts]

"$\Rightarrow$" We have to prove that if (z_n) converges, then (x_n) and (y_n) converge to the same limit.

The sequences (x_n) and (y_n) are subsequence of (z_n) (taking respectively even and odd indices). Thus, if (z_n) converges, the subsequence should converge as well and to the same limit. 1.5pt

"$\Leftarrow$" Suppose now $x_n \to l$ and $y_n \to l$. We have to show that $z_n \to l$ as well. 1.5pt

Fix $\varepsilon > 0$. Consider N_1 and N_2 such that $|x_n - l| < \varepsilon$ for $n \geq N_1$ and $|y_n - l| < \varepsilon$ for $n \geq N_2$. Consider now $N = \max(\mathbf{2}N_1, \mathbf{2}N_2 + 1)$. Then, for any $n \geq N$, we have either $z_n = x_{n_k}$ with $n_k \geq N_1$ or $z_n = y_{n_k}$ with $n_k \geq N_2$. In both cases, $|z_n - l| < \varepsilon$.

Ex 9.

a) We have to show that $a_n \cdot b_n \to 0$ when a_n bounded and $b_n \to 0$.

   Fix $\varepsilon > 0$. Since a_n bounded, there exists M such that $|a_n| \leq M$ for any n. Consider N such that $|b_n - 0| < \varepsilon/M$ for $n \geq N$. For any $n \geq N$, we have:

   $$|a_n \cdot b_n - 0| \leq M|b_n| < M\frac{\varepsilon}{M} = \varepsilon.$$

   We cannot use the Algebraic Limit Theorem since the sequence (a_n) is not converging.

b) If we only have $b_n \to l$ with l not zero, then the sequence $(a_n \cdot b_n)$ does not necessarily converges. For instance $a_n = (-1)^n$ and $b_n = 1$. However, we know that the sequence $(a_n \cdot b_n)$ is bounded (since (a_n) and (b_n) are bounded).

c) If we assume that $a_n \to 0$, to prove the theorem, we use that since (b_n) convergent it is also bounded. Then we can use a) (changing (a_n) and (b_n)) to conclude.

Ex 11.

a) Suppose $x_n \to l$. We have to show that $y_n = \frac{1}{n}(x_1 + \cdots + x_n)$ also converges to l.

Fix $\varepsilon > 0$, and consider N such that $|x_n - l| < \varepsilon/2$. Take now $n \geq N$, we need to control:

$$\begin{aligned} |y_n - l| &= \frac{1}{n}\left|(x_1 - l) + \cdots + (x_n - l)\right| \\ &= \frac{1}{n}\left|(x_1 - l) + \cdots + (x_N - l) \ + \ (x_{N+1} - l) + \cdots + (x_n - l)\right| \\ &\leq \frac{1}{n}\left|(x_1 - l) + \cdots + (x_N - l)\right| \ + \ \frac{1}{n}\left|(x_{N+1} - l) + \cdots + (x_n - l)\right| \end{aligned}$$

We have two parts to control. For the first part of the sum, let $M = |(x_1 - l) + \cdots + (x_N - l)|$. Consider N_2 such that $M/N_2 < \varepsilon/2$. Thus, if $n \geq N_2$, we have:

$$\frac{1}{n}\left|(x_1 - l) + \cdots + (x_N - l)\right| \ < \ \varepsilon/2.$$

For the second part, we have:

$$\begin{aligned} \frac{1}{n}\left|(x_{N+1} - l) + \cdots + (x_n - l)\right| &\leq \frac{1}{n}(|x_{N+1} - l| + \cdots + |x_n - l| \\ &< \frac{1}{n}(\varepsilon/2 + \cdots + \varepsilon/2) \ = \ \frac{n-N}{n}\varepsilon/2 < \varepsilon/2. \end{aligned}$$

We conclude that for $n \geq \max(N, N_2)$, we have $|y_n - l| < \varepsilon$.

b) Consider the sequence $x_n = (-1)^n$. The sequence is not converging. However, the Cesaro means gives:

$$y_n = \frac{1}{n}(x_1 + \cdots + x_n) = \begin{cases} 1/n & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

It is clear that $y_n \to 0$.

3 Chapter 2.4

Ex 1. [4pts]

a) Computing the first values of the sequence gives: $x_1 = 3$, $x_2 = 1$, $x_3 = 1/3$. Thus, we would like to show that the sequence is decreasing and lower bounded by 0 to apply the monotone convergence theorem.

We prove by induction that $0 \leq x_n$ and that $x_n < x_{n-1} < \cdots < x_1$. The result is clearly true for $n = 2$. We suppose it is true up to N and we would like to show that it is still true at $N + 1$. We have:

$$x_{N+1} = \frac{1}{4 - x_N}$$

Since $x_N > 0$, we also have $x_{N+1} > 0$. We can in particular take its inverse:

$$\frac{1}{x_{N+1}} = 4 - x_N$$

We can use the assumption that $x_N < x_{N-1}$:

$$\frac{1}{x_{N+1}} = 4 - x_N > 4 - x_{N-1} = \frac{1}{x_N}.$$ 1pt

Therefore $x_{N+1} < x_N$. We conclude that $0 < x_n$ and $x_{n+1} < x_n$ for all n.

We conclude using the monotone convergence theorem that the sequence (x_n) converges to some l in $\mathbb{R}$. 1pt

b) Fix $\varepsilon > 0$, since $x_n \to l$, there exists N such that $|x_n - l| < \varepsilon$ for $n \geq N$. This also implies that $|x_{n+1} - l| < \varepsilon$. Thus, (x_{n+1}) converges to the same limit l. 1pt

c) Passing in the limit $n \to +\infty$ in the expression gives:

$$l = \frac{1}{4-l} \quad \Rightarrow \quad (4-l)l = 1 \quad \Rightarrow \quad l^2 - 4l + 1 = 0.$$

We find two solutions $l = 2 \pm \sqrt{3}$. Since the sequence (x_n) is decaying, its limit l has to be smaller than $x_1 = 3$. Thus, $l = 2 - \sqrt{3}$. 1pt