MAT 371: Homework 5 (02/12)

1 Chapter 2.5

Ex 1.

a) Not possible: if there exists a subsequence (a_{n_k}) bounded, then there is a subsequence (of this subsequence) that would be convergence.

b) Take $a_n = \frac{1}{10+n} + \frac{1}{2}(1 + (-1)^n)$. We have $a_n \neq 0$ and $a_n \neq 1$ for all n. But $a_{2n} \to 1$ and $a_{2n+1} \to 0$.

c) Take a sequence that is 'dense' in $[0, 1]$. We know that $\mathbb{N} \sim \mathbb{Q}$, thus we can find a sequence that covers *only* $\mathbb{Q} \cap [0, 1]$.

Ex 5.

We proceed by contradiction: assume that (a_n) does *not* converge to a. This means that there exists $\varepsilon > 0$, such that:

$$\text{for any } N, \text{ there exists } n \geq N \text{ satisfying } |a_n - a| \geq \varepsilon. \tag{1}$$

The idea is now to construct a subsequence (a_{n_k}) that will satisfy $|a_{n_k} - a| \geq \varepsilon$ for all n_k. Applying (1) with $N = 1$, there exists $n_1 \geq 1$, such that $|a_{n_1} - a| \geq \varepsilon$. Take now $N = n_1 + 1$ and apply (1) once more: there exists $n_2 \geq n_1 + 1$ such that $|a_{n_2} - a| \geq \varepsilon$. We can iterate this process to construct a subsequence (a_{n_k}) such that $|a_{n_k} - a| \geq \varepsilon$ for all n_k.

From this subsequence (a_{n_k}), we can extract a sub(-sub)sequence that converges. Indeed, since (a_n) bounded, the subsequence (a_{n_k}) is also bounded and we can extract a converging subsequence by Bolzano-Weierstrass. We denote by m_k the new indices of this sequence (e.g. $n_1 = 2$, $n_2 = 4$, $n_3 = 6\ldots$, then we may have $m_1 = 2$, $m_2 = 6\ldots$). It still satisfies $|a_{m_k} - a| \geq \varepsilon$. By assumption, this convergent subsequence has to converge to a: $a_{m_k} \to a$. Therefore, there should exist N such that $|a_{m_k} - a| < \varepsilon$ for $m_K \geq N$. Contradiction.

2 Chapter 2.6

Ex 2.

a) Take (a_n) convergent and not monotone. For example $a_n = (-1)^n/n$.

b) Not possible: if (a_n) Cauchy, then it is bounded. Therefore, all of its sub-sequences will be bounded too.

c) Not possible: we proceed by contradiction. The sequence (a_n) is not bounded (otherwise it will converge by the monotone convergence theorem). Thus, we either have $a_n \to +\infty$ (a_n increasing) or $a_n \to -\infty$ (if a_n decreasing). Let's suppose (a_n) is increasing, the argument is similar with (a_n) decreasing. Thus, for any M, there exists n such that $a_n > M$.

Suppose now that there exists a Cauchy subsequence (a_{n_k}), in particular this subsequence has to converge (Cauchy implies CV in $\mathbb{R}$): there exists a such that $a_{n_k} \to a$. Fix $\varepsilon = 1$, there exists N such that:

$$|a_{n_k} - a| < 1 \quad \text{for all } n_k > N.$$

Since (a_n) is unbounded, there exists m such that $a_m > a + 10$. Take $n_k > m$, since (a_n) is increasing, we must have $a_{n_k} \geq a_m > a + 10$. Contradiction.

d) Take $a_n = \begin{cases} n & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$ Thus, $a_0 = 0,\ a_1 = 1,\ a_2 = 0,\ a_3 = 3,\ a_4 = 0,\ a_5 = 5\ldots.$
The sequence is unbounded, by the subsequence (a_{n_k}) with n_k even number is a Cauchy sequence.

Ex 4.

a) Denote a and b the limits of the two sequences: $a_n \to a$ and $b_n \to b$. Let $c_n = |a_n - b_n|$. We have $c_n \to |a - b|$ thus (c_n) is a Cauchy sequence since it converges.

b) Not Cauchy in general: take $a_n = 1$ (Cauchy sequence), we have $c_n = (-1)^n$ not converging, therefore not Cauchy.

c) Not Cauchy in general: take $a_n = 10 + .2 \cdot (-1)^n/n$ which gives $a_1 = 9.8, a_2 = 10.1,\ a_3 = 9.9\ldots,\ a_4 = 10.0\ldots,$. Therefore $c_1 = 9,\ c_2 = 10,\ c_3 = 9\ldots.$ The sequence (c_n) oscillates between 9 and 10 and does not converge.

Remark. For part c), (c_n) is (in general) not Cauchy because the function $x \to [[x]]$ is *not* continuous.

3 Chapter 2.7

Ex 2.

a) Converging: $0 \leq a_n \leq \frac{1}{2^n}$ and $\sum \frac{1}{2^n}$ converges.

b) Converging: $|a_n| \leq \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges.

c) Not converging: $a_n = \frac{(-1)^n(n-1)}{2n}$ but a_n does not converge to zero. Thus, the series cannot converge.

d) Not converging:

$$\begin{aligned} 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots &\geq 1 + 0 - 0 + \frac{1}{4} + 0 - 0 \dots \\ &\geq \sum_{n=0}^{+\infty} \frac{1}{3n+1} \\ &\geq \frac{1}{3} \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty. \end{aligned}$$

e) Not converging: let $a_n = 1/(2n+1)$ and $b_n = -1/(2(n+1))^2$. We have:

$$1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \dots = a_0 - b_0 + a_1 - b_1 \dots = \sum_{n=0}^{+\infty} (a_n - b_n).$$

Since $\sum a_n$ not converging and $\sum b_n$ converging, we deduce that $\sum(a_n - b_n)$ not converging.