# MAT 371: Midterm 1 Feb. 14, 2019 Duration: 1:15

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**Exercise 1.** 20pts

We have $\sup A = 1$. Indeed, for $n \geq 0$, $n/(n+1) < 1$, therefore 1 is an upper-bound. 4+3pts

Moreover, for any $\varepsilon > 0$, there exists $n$ such that $1/n < 1/\varepsilon$ and therefore $1-\varepsilon < 1-\frac{1}{n}$. In other words, there exists $x \in A$ such that $\sup A - \varepsilon < x$. Thus, 1 is the lowest upper-bound. 4pts

The inf is given by $\inf A = 0$. Indeed, $n/(n+1) \geq 0$ thus 0 is a lower-bound. Moreover, $0 \in A$, thus it is necessarily the largest lower-bound. 4+2pts 3pts

**Exercise 2.** 25pts

a) For any $\varepsilon > 0$, there exists $N$ such that $|a_n - a| < \varepsilon$ for any $n \geq N$. 6pts

b) Let's do the preliminary computation: 4pts

$$|a_n b_n - ab| \;=\; |(a_n - a)b_n + a(b_n - b)| \leq |a_n - a| \cdot |b_n| + |a||b_n - b|.$$

Since $(b_n)$ converges and it is also bounded: there exists $M$ such that $|b_n| \leq M$ for any $n$. Denote $C = \max(M, |a|)$. The previous inequality reads: 4pts

$$|a_n b_n - ab| \leq C(|a_n - a| + |b_n - b|).$$

Fix now $\varepsilon > 0$, since $(a_n)$ and $(b_n)$ converge there exists $N$ such that: 4pts

$$|a_n - a| < \frac{\varepsilon}{2C} \quad \text{and} \quad |b_n - b| < \frac{\varepsilon}{2C} \qquad \text{for } n \geq N.$$

We then deduce that:

$$|a_n b_n - ab| < C\left(\frac{\varepsilon}{2C} + \frac{\varepsilon}{2C}\right) = \varepsilon.$$

c) Take $a_n = n$ and $b_n = \frac{1}{n}$, the product $(a_n \cdot b_n)$ converges but not $(a_n)$. 3pts

Another example, take: 4pts

$$a_n = \begin{cases} n & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases} \quad , \quad b_n = \begin{cases} 0 & \text{if } n \text{ even} \\ n & \text{if } n \text{ odd} \end{cases}$$

Both sequences $(a_n)$ and $(b_n)$ diverge, but $a_n \cdot b_n = 0$ for all $n$ thus converges.

**Exercise 3.** 30pts

Let $a_0 = 0$ and $a_1 = 4$. We consider the recursive sequence: $a_{n+2} = \frac{a_n + a_{n+1}}{2}$.

a) $a_2 = 2$, $a_3 = 3$, $a_4 = 2.5$. 6pts

b) We use that $a_{n+1} = \frac{a_{n-1}+a_n}{2}$:

$$|a_{n+1} - a_n| = \left|\frac{a_{n-1} + a_n}{2} - a_n\right| = \left|\frac{a_{n-1} - a_n}{2}\right|.$$ 4pts

We deduce by iteration:

$$|a_{n+1} - a_n| = \frac{1}{2}|a_n - a_{n-1}| = \left(\frac{1}{2}\right)^2 |a_{n-1} - a_{n-2}| = \ldots = \left(\frac{1}{2}\right)^n |a_1 - a_0|.$$ 3pts

c) Fix $m > n$. We deduce:

$$\begin{aligned} |a_n - a_m| &= |a_n - a_{n+1} + a_{n+1} - a_{n+2} + \cdots + a_{m-1} - a_m| \\ &\leq |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \cdots + |a_{m-1} - a_m| \\ &= \left(\frac{1}{2}\right)^n |a_1 - a_0| + \left(\frac{1}{2}\right)^{n+1} |a_1 - a_0| + \ldots + \left(\frac{1}{2}\right)^m |a_1 - a_0| \\ &= \left(\frac{1}{2}\right)^n |a_1 - a_0| \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-1-n}}\right). \end{aligned}$$

4pts

3pts

d) We deduce from *c*) that:

$$\begin{aligned} |a_n - a_m| &\leq \left(\frac{1}{2}\right)^n |a_1 - a_0| \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-1-n}}\right) \\ &\leq \left(\frac{1}{2}\right)^n |a_1 - a_0| \cdot \sum_{k=0}^{+\infty} \frac{1}{2^k} = \left(\frac{1}{2}\right)^n |a_1 - a_0| \cdot \frac{1}{1 - 1/2} \\ &\leq \left(\frac{1}{2}\right)^{n-1} |a_1 - a_0| \overset{n\to+\infty}{\longrightarrow} 0. \end{aligned}$$

3+3pts

Thus, $(a_n)$ satisfies the Cauchy criteria. It converges in $\mathbb{R}$. 4pts

*Extra)* To find the limit of $(a_n)$, we have to observe a *pattern* (see figure ):

$$\begin{aligned} a_2 &= 4 - 2 \\ a_3 &= 4 - 2 + 1 \\ a_4 &= 4 - 2 + 1 - \frac{1}{2} \end{aligned}$$

Thus,

$$a_n = 4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} + \ldots + \left(-\frac{1}{2}\right)^{n-3}.$$

Taking the limit $n \to \infty$ gives:

$$a = 2 + \sum_{k=0}^{+\infty} \left(-\frac{1}{2}\right)^k = 2 + \frac{1}{1 - (-1/2)} = 2 + \frac{2}{3}.$$

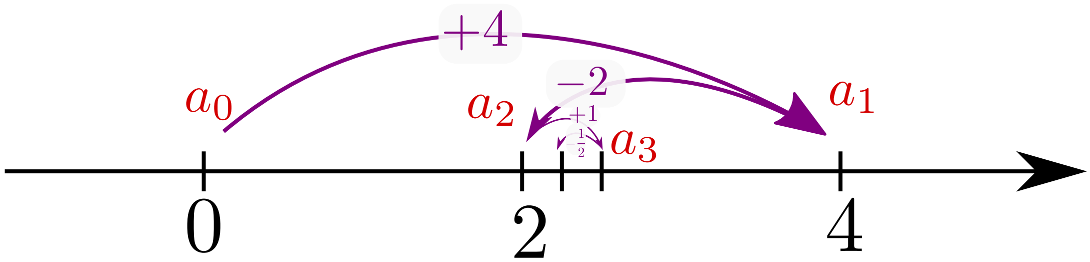


Figure 1: The sequence $(a_n)$ can be seen as a partial sum.

**Exercise 4.** 25pts

a) Fix $\varepsilon > 0$. Since $\sum_{n=0}^{+\infty} a_n$ converges, we can use the Cauchy criteria: there exists $N$ such for any $m \geq n \geq N$ we have:

4pts

$$|a_{n+1} + \cdots + a_m| < \varepsilon.$$

Taking $m = n + 1$ gives simply $|a_{n+1}| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary this proves that $(a_n)$ converges to zero. 3pts

b) We have $a_n = \frac{n+1}{10 \cdot n+5} \to \frac{1}{10}$. Since $(a_n)$ does not converge to zero, the series cannot converge. 4+3pts

c) Fix $\varepsilon > 0$. Since $(b_n)$ is supposed bounded, let $M$ an upper bound of $(b_n)$. The series $\sum a_n$ converges, thus the Cauchy criteria gives that there exists $N$ such that for any $m \geq n \geq N$:

$$|a_{n+1} + \cdots + a_m| < \varepsilon/M.$$

4pts

Since $(a_n)$ are all positive, we deduce: $a_{n+1} + \cdots + a_m < \varepsilon/M$. Therefore,

$$a_{n+1}b_{n+1} + \cdots + a_m b_m \leq a_{n+1}M + \cdots + a_m M \leq (a_{n+1} + \cdots + a_m)M < \varepsilon.$$

Since $(a_n b_n)$ are all positive, we conclude that: 2pts

$$|a_{n+1}b_{n+1} + \cdots + a_m b_m| < \varepsilon \quad \text{for all } m \geq n \geq N.$$

Since the series $\sum a_n b_n$ satisfies the Cauchy criteria, it converges.

d) The previous result does not hold if we do not suppose both sequence positives. Take $a_n = (-1)^n/n$ and $b_n = (-1)^n$. The series $\sum a_n$ converges and $(b_n)$ is bounded. However, $a_n b_n = 1/n$ gives a series that do not converge. 5pts