MAT 371: Practice midterm exam

Exercise 1. **[inf/sup]**

Suppose A bounded set in $\mathbb{R}$. Denote sup$A$ the supremum of A (the lowest upper-bound of A)

a) Since $\sup A - \varepsilon$ is not an upper-bound, there exists $x \in A$ such that $x > \sup A - \varepsilon$. Moreover, since $\sup A$ is an upper-bound for A, we also have $x \leq \sup A$. Combining these two results give $\sup A - \varepsilon < x \leq \sup A$.

b) The result is not true when A is 'discrete'. For instance, take $A = \{0, 1, 2, 3\}$, we have $\sup A = 3$. For $\varepsilon = \frac{1}{2}$, we will not find any points in A that are *stricly* in between $\sup A - \varepsilon$ and $\sup A$.

Exercise 2. **[cardinality]**

a) Take $f(x) = 1/x - 1/\varepsilon$.

b) Consider $f(n) = \begin{cases} k & \text{if } n = 2k \\ -k & \text{if } n = 2k+1 \end{cases}$

c) The trick is to cover $\mathbb{N} \times \mathbb{N}$ using the diagonals (see figure 1) step by step. For instance, we have:

$$f(0) = (0,0),\ f(1) = (1,0),\ f(2) = (0,1),\ f(3) = (2,0)\ldots$$

It is clear that for any $(i,j) \in \mathbb{N} \times \mathbb{N}$, there exists (a unique) $k \in \mathbb{N}$ such that $f(k) = (i,j)$.

Exercise 3. **[convergence]**

a) For any $\varepsilon > 0$, there exists N (which depends on ε) such that:

$$|a_n - a| < \varepsilon \quad \text{for any } n \geq N.$$

b) Fix $\varepsilon > 0$. Take $N > 1/(3\varepsilon)$. For any $n \geq N$, we have:

$$\begin{aligned} |a_n - 2| &= \left|\frac{2n+1}{n+2} - 2\right| = \left|\frac{-3}{n+2}\right| \\ &\leq \frac{3}{n} \leq \frac{3}{N} < \varepsilon. \end{aligned}$$

c) Let $a_n = 2 + (-1)^n$. Take $\varepsilon = \frac{1}{2}$. For any N, there exists n odd integer such that $n \geq N$. In this case, we have $|a_n - 2| = |1 - 2| = 1 > \varepsilon$. Therefore, (a_n) cannot converge to 2.
To show that the sequence (a_n) does not converge, we can show that (a_n) does not satisfy Cauchy criteria (taking $\varepsilon = 1/2$ again).

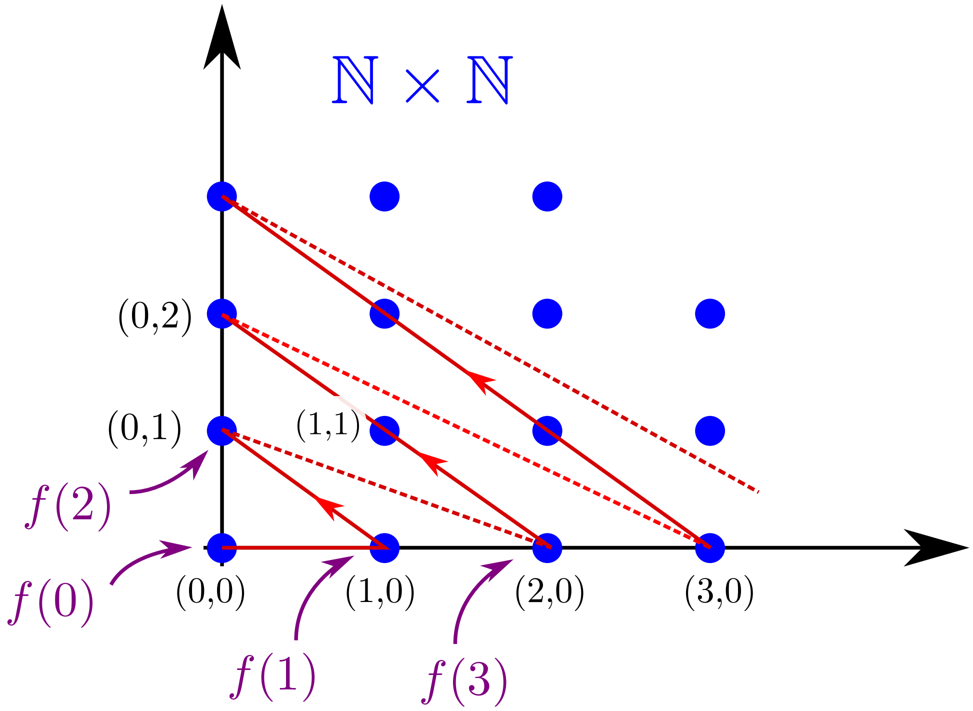

Figure 1: Illustration of a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.

Exercise 4.

a) Fix $\varepsilon > 0$. Consider N_1 such that $|a_n - a| < \varepsilon/20$ for $n \geq N_1$ and N_2 such that $|b_n - b| < \varepsilon/20$ for $n \geq N_2$. Take now $N = \max(N_1, N_2)$. For any $n \geq N$, we have:

$$\begin{aligned} |5a_n + 10b_n - (5a + 10b)| &= |5(a_n - a) + 10(b_n - b)| \leq 5|a_n - a| + 10|b_n - b| \\ &< 5\frac{\varepsilon}{20} + 10\frac{\varepsilon}{20} < \varepsilon. \end{aligned}$$

b) Take $\varepsilon = 1$, there exists N such that for any $n \geq N$, we have $|a_n - a| < 1$. In particular, we deduce (triangular inequality):

$$|a_n| \leq |a| + 1 \quad \text{for } n \geq N.$$

Take now $M = \max(|a_0|, |a_1|, \ldots, |a_{N-1}|, |a| + 1)$. For any n, we have $|a_n| \leq M$.

Exercise 5. [monotone convergence]
Consider the sequence defined recursively with $a_0 = 10$ and

$$a_{n+1} = \sqrt{2a_n + 5}.$$

a) $a_1 = \sqrt{25} = 5$, $a_2 = \sqrt{15}$.

b) We are going to show by induction that $0 \leq a_{n+1} < a_n$.

It is clear that for $n = 0$, we do have $0 \leq a_1 < a_0$. Suppose (induction hypothesis) that the inequality is true up to N, i.e. $0 \leq a_{n+1} < a_n$ for any $0 \leq n \leq N$, we are going that it is still valid for $N+1$. By hypothesis of induction, we know that:

$$\begin{aligned} 0 \le a_{N+1} < a_N \quad &\Rightarrow \quad 0 \le 2a_{N+1}+5 < 2a_N+5 \\ &\Rightarrow \quad 0 \le \sqrt{2a_{N+1}+5} < \sqrt{2a_N+5} \\ &\Rightarrow \quad 0 \le a_{N+2} < a_{N+1}, \end{aligned}$$

where we use that the function $x \to \sqrt{x}$ is increasing. Thus, the induction hypothesis is still valid up to $N+1$. We conclude that $0 \le a_{n+1} \le a_n$ is valid for any n.

c) Since the sequence (a_n) is decreasing and lower-bounded, we deduce that the sequence converges (monotone theorem): there exists a such that $a_n \to a$.

   To find its limit, we pass to the limit in the equation: $a_{n+1} = \sqrt{2a_n+5}$. We deduce that:

$$a = \sqrt{2a+5}$$

   Therefore $a^2 = 2a+5$. Solving the equation $a^2-2a-5=0$ leads to: $a = 1\pm\sqrt{1+5}$. Since $a_n \ge 0$, we must also have $a \ge 0$. Therefore, $a = 1+\sqrt{6}$.

Exercise 6. [**subsequence**]

Fix $\varepsilon > 0$. Since $a_n \to a$, there exists N such that $|a_n - a| < \varepsilon$ for any $n \ge N$. We deduce that for any $n_k \ge N$, we also have: $|a_{n_k} - a| < \varepsilon$. Thus, $a_{n_k} \to a$ as well.

Taking the subsequence $n_k = 2k$ (i.e. $n_1 = 2$, $n_2 = 4$, $n_3 = 6\ldots$), leads to $a_{n_k} = 1$ and thus $a_{n_k} \to 1$. Similarly, with $m_k = 2k+1$, we obtain $a_{m_k} = -1 \to -1$. Thus, two sub-sequences of (a_n) converges to two different limit. Therefore, the sequence (a_n) cannot converge.

Exercise 7. [**Cauchy criterion**]

a) For any $\varepsilon > 0$, there exists N such that:

$$|a_n - a_m| < \varepsilon \quad \text{for any } m \ge n \ge N. \tag{1}$$

b) Fix $\varepsilon = 1$. Take N such that (1) is satisfied. In particular, taking $n = N$, we have for any $m \ge N$:

$$|a_N - a_m| < 1,$$

   Therefore $|a_m| \le |a_N| + 1$. Let $M = \max(|a_0|, |a_1|, \ldots, |a_{N-1}|, |a_N|+1)$, we deduce that $|a_n| \le M$ for any n.

Exercise 8. [**Infinite series**]

a) We are going to use the Cauchy criteria. Fix $\varepsilon > 0$. There exists N such that:

$$|b_{n+1} + \cdots + b_m| < \varepsilon \quad \text{for all } m \ge n \ge N.$$

   Since all terms are positive (i.e. $b_n \ge 0$ for all n), we also have:

$$0 \le b_{n+1} + \cdots + b_m < \varepsilon$$

Using that $a_n \leq b_n$, we deduce:

$$a_{n+1} + \cdots + a_m \leq b_{n+1} + \cdots + b_m.$$

Since $a_n \geq 0$, we conclude that:

$$|a_{n+1} + \cdots + a_m| \leq |b_{n+1} + \cdots + b_m|.$$

Therefore $|a_{n+1} + \cdots + a_m| < \varepsilon$. The series $\sum a_n$ satisfies the Cauchy criteria and therefore converges.

b) Let $a_n = \frac{10^n}{n!}$:

- We have:

$$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}}{(n+1)!} \frac{n!}{10^n} = \frac{10}{(n+1)}.$$

  Thus for $n \geq 20$, we deduce:

$$\frac{a_{n+1}}{a_n} < \frac{10}{20} = \frac{1}{2}.$$

- We deduce by iteration that:

$$a_{20+n} \leq \frac{1}{2} a_{20+n-1} \leq \left(\frac{1}{2}\right)^2 a_{20+n-2} \leq \ldots \leq \left(\frac{1}{2}\right)^n a_{20+n-n} = \left(\frac{1}{2}\right)^n a_{20}.$$

- We decompose the series in two parts:

$$\sum_{n=0}^{+\infty} a_n = \sum_{n=0}^{20} a_n + \sum_{n=21}^{+\infty} a_n$$

  The first sum is finite (finite number of elements) and for the second part we use the previous part:

$$\sum_{n=21}^{+\infty} a_n \leq \sum_{n=21}^{+\infty} a_{20} \left(\frac{1}{2}\right)^n < +\infty$$

  since it is a geometric series.