MAT 371: Homework 10 (04/16)

1 Chapter 6.2

Ex 1. [4pts]

a) For any $x \in (0,+\infty)$, we have:

$$f_n(x) = \frac{nx}{1+nx^2} = \frac{x}{1/n + x^2} \overset{n\to+\infty}{\longrightarrow} \frac{x}{x^2} = \frac{1}{x}.$$

Indeed, let's denote $f(x) = 1/x$ define for $x \in (0,+\infty)$, we have:

$$|f_n(x) - f(x)| = \left|\frac{nx}{1+nx^2} - \frac{1}{x}\right| = \left|\frac{1}{x(1+nx^2)}\right|$$

Therefore, if $x > 0$,

1pt

$$|f_n(x) - f(x)| \le \left|\frac{1}{x \cdot nx^2}\right| = \frac{1}{n}\left|\frac{1}{x^3}\right| \quad \overset{n\to+\infty}{\longrightarrow} \quad 0.$$

b) The convergence is however not uniform. Indeed, when x is close to zero, the difference between f_n and f *blows-up*. Fix n, we have: 1pt

$$|f_n(x) - f(x)| \left|\frac{1}{x(1+nx^2)}\right| \overset{x\to 0}{\longrightarrow} +\infty.$$

Thus, $\sup_{x\in(0,+\infty)} |f_n(x) - f(x)| = +\infty$ for all n and therefore does not converge to zero as $n \to +\infty$.

c) Similarly, we have $\sup_{x\in(0,1)} |f_n(x) - f(x)| = +\infty$, thus the convergence is not uniform either on $(0,1)$. 1pt

d) However, on $I = [1,+\infty)$, we have: 1pt

$$|f_n(x) - f(x)| \left|\frac{1}{x(1+nx^2)}\right| \le \left|\frac{1}{nx^3}\right| \le \left|\frac{1}{n}\right|$$

since $x \ge 1$. Therefore, we have a uniform bound on the error (independent of x): $\sup_{x\in(1,+\infty)} |f_n(x) - f(x)| \le 1/n \overset{n\to+\infty}{\longrightarrow} 0$. We deduce that the convergence is uniform on $[1,+\infty)$.

Ex 3.

a) The pointwise limit is given by:

$$g_n(x) = \frac{x}{1+x^n} \overset{n\to+\infty}{\longrightarrow} g(x) = \begin{cases} x & \text{if } x < 1 \\ x/2 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

$$h_n(x) = \begin{cases} 1 & \text{if } x > 1/n \\ nx & \text{if } 0 \le x < 1/n \end{cases} \overset{n\to+\infty}{\longrightarrow} h(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

b) The functions f_n and g_n are all continuous. If the convergence were uniform, the limit functions would be also continuous and there are not.

c) For h, the convergence will be uniform on any interval of the form $I = [\varepsilon, \infty)$ with $\varepsilon > 0$. Indeed, for n large enough we will have $h_n(x) = 1$ on I and therefore $\sup_I |h_n(x) - h(x)| = 0$.

For g, we can consider an interval of the form $I = [1+\varepsilon, +\infty)$. Indeed, we can find a uniform bound (independent of x)

$$|g_n(x) - g(x)| = \left|\frac{x}{1+x^n} - x\right| = \left|\frac{x^{n+1}}{1+x^n}\right| \le \frac{(1+\varepsilon)^{n+1}}{1+1}$$

and this bound converges to zero as $n \to +\infty$.

Ex 9.

We assume that $f_n \to f$ and $g_n \to g$ both uniformly on some interval I.

a) Taking the difference:

$$\begin{aligned} |f_n(x) + g_n(x) - (f(x) + g(x)| &= |f_n(x) - f(x) + g_n(x) - g(x)| \\ &\le |f_n(x) - f(x)| + |g_n(x) - g(x)|. \end{aligned}$$

We deduce that:

$$\sup_I |f_n + g_n - (f+g)| \le \sup_I |f_n - f| + \sup_I |g_n - g| \overset{n\to+\infty}{\longrightarrow} 0.$$

Thus, $f_n + g_n$ converges uniform to $f + g$ on I.

b) Consider $f_n(x) = x$ (independent of n) and $g_n(x) = 1/n$. We have uniform convergence on $\mathbb{R}$ to $f(x) = x$ and $g(x) = 0$. However, $f_n(x)g_n(x) = x/n$ does not converge uniform to 0 on $\mathbb{R}$ since:

$$\sup_{x\in\mathbb{R}} |f_n(x)g_n(x) - 0| = \sup_{x\in\mathbb{R}} |x/n| = +\infty.$$

c) Assuming now f_n and g_n are both bounded on I by some constant M (we say that there are *uniformly* bounded since M does not depend on n). We have:

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &= |(f_n(x) - f(x))g_n(x) \;+\; f(x)(g_n(x) - g(x))| \\ &\le M|f_n(x) - f(x)| \;+\; M|g_n(x) - g(x)| \\ &\le M(\sup_I |f_n - f| + \sup_I |g_n - g|) \overset{n\to+\infty}{\longrightarrow} 0. \end{aligned}$$

Therefore, the convergence is uniform.

2 Chapter 6.3

Ex 1. [3pts]

a) For any $x \in [0,1]$, $g_n(x)$ converges uniformly to $g(x) = 0$. Indeed, since $|x| \leq 1$, [1.5pt]

$$|g_n(x) - g(x)| = \left|\frac{x^n}{n}\right| \leq 1/n \overset{n\to+\infty}{\longrightarrow} 0.$$

The function g is clearly differentiable with $g'(x) = 0$.

b) We have $g_n'(x) = x^{n-1}$. We have seen that the sequence of function convergences pointwise but not uniformly to: [1.5pt]

$$g_n'(x) = x^{n-1} \overset{n\to+\infty}{\longrightarrow} h(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

In particular, we find that $h \neq g'$ at $x = 1$ which is not a surprise as the convergence of g_n' is not uniform.

Ex 2.

a) We find that $h_n(x) \overset{n\to+\infty}{\longrightarrow} \sqrt{x^2} = |x|$. To show that the convergence is uniform, denote $h(x) = |x|$ and consider the difference:

$$|h_n(x) - h(x)| \quad = \quad \left|\sqrt{x^2 + 1/n} - |x|\right|.$$

To remove the square root, we can use the trick $(a - b) = (a - b) \cdot \frac{a+b}{a+b} = \frac{a^2-b^2}{a+b}$ leading to:

$$|h_n(x) - h(x)| \quad = \quad \left|\frac{\sqrt{x^2+1/n}^2 - |x|^2}{\sqrt{x^2+1/n} + |x|}\right| = \left|\frac{1/n}{\sqrt{x^2+1/n} + |x|}\right|.$$

Finally,

$$|h_n(x) - h(x)| \leq \left|\frac{1/n}{\sqrt{1/n + 0}}\right| = \sqrt{1/n} \overset{n\to+\infty}{\longrightarrow} 0.$$

b) Computing the differential of g_n' gives:

$$h_n'(x) = \frac{1}{2}(2x)(x^2 + 1/n)^{-1/2} = \frac{x}{\sqrt{x^2 + 1/n}}$$

which is a continuous function on $\mathbb{R}$ for any n. Since the limit function $h(x) = |x|$ is not differentiable at $x = 0$, the convergence of h_n' cannot be uniform otherwise near $x = 0$ otherwise h will be also differentiable.

Ex 3. [3pts]

a) Since the function f_n goes to zero at infinity (i.e. $\lim_{x\to\pm+\infty} f(x) = 0$), the minimum and maximum values of f are reached *inside* $\mathbb{R}$. Moreover, since f_n is smooth, the min/max are critical point. Thus, we look for solution to $f_n'(x) = 0$. Let's first compute f_n':

$$f_n'(x) = \frac{1\cdot(1+nx^2) - x\cdot 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}.$$ 1pt

Thus, solving $f_n'(x) = 0$ leads to:

$$1 - nx^2 = 0 \quad \Rightarrow \quad x^2 = 1/n \quad \Rightarrow \quad x = \pm 1/\sqrt{n}.$$ 1pt

Computing the values of f_n at these two critical points, we see that f_n is maximum at $1/\sqrt{n}$ and minimum at $-1/\sqrt{n}$.

b) For any $x \neq 0$, we have:

$$|f_n'(x)| \leq \left|\frac{1-nx^2}{(nx^2)^2}\right| \leq \left|\frac{1}{(nx^2)^2}\right| + \left|\frac{nx^2}{(nx^2)^2}\right| \overset{n\to+\infty}{\longrightarrow} 0.$$ 1pt

However, for $x = 0$, we have: $f_n'(0) = 1 \overset{n\to+\infty}{\longrightarrow} 1$.
We deduce that f_n convergence uniformly to $f = 0$ but f_n' does not converge uniformly to f'.

3 Chapter 6.4

Ex 2.

a) True: apply the Cauchy criteria with $m = n + 1$.

b) True: using Cauchy criteria. Fix $\varepsilon > 0$, by uniform convergence of $\sum g_n$, there exists $N > 0$ such that:

$$\sup_I |g_m(x) + \cdots + g_{n+1}(x)| < \varepsilon \qquad \text{for } m \geq n \geq N.$$

Moreover, since $0 \leq f_n(x) \leq g_n(x)$, we have:

$$\begin{aligned} |f_m(x) + \cdots + f_{n+1}(x)| &= f_m(x) + \cdots + f_{n+1}(x) \leq g_m(x) + \cdots + g_{n+1}(x) \\ &\leq |g_m(x) + \cdots + g_{n+1}(x)|. \end{aligned}$$

Therefore, we also have

$$\sup_I |f_m(x) + \cdots + f_{n+1}(x)| < \varepsilon \qquad \text{for } m \geq n \geq N$$

and we deduce that $\sum f_n$ also satisfies the Cauchy criteria.
Notice that the result is not true if we do not suppose $0 \leq f_n(x)$.

c) False: Consider $f_n(x) = \frac{(-1)^n}{n}x$ on $A = [0,1]$. The series $\sum_{n\geq 1} f_n$ converges uniformly on A. But $M_n = \sup_A |f_n(x)| = 1/n$ and $\sum_n M_n = +\infty$.
In other words, the Weierstrass criteria used for proving uniform convergence of series is a *sufficient* condition but not a *necessarily* condition.