MAT 371: Homework 11 (04/23)

1 Chapter 6.5

Ex 2. [4pts]

a) Take $a_n = 0$ or $a_n = 1/n!$ (leading to the exponential function). 1pt

b) Not possible: at $x = 0$ the sum is always equal to a_0. 1pt

c) Take $a_n = 1/n^2$. 1pt

d) Not possible: if the converge is absolute at $x = 1$ $\sum |a_n(1)^n| < +\infty$, then $\sum |a_n(-1)^n| < +\infty$ as well and therefore the converge is also absolute at $x = -1$. 1pt

e) We need to find a sequence (a_n) such that $a_n \cdot 1$ and $a_n \cdot (-1)^n$ will both 'oscillate'. Consider:

$$a_1 = 1,\ a_2 = 1/2,\ a_3 = -1/3,\ a_4 = -1/4,\ a_5 = 1/5,\ a_6 = 1/6,\ \ldots$$

in other words:

$$a_{3n+k} = \begin{cases} \frac{1}{4n+k} & \text{if } k = 1 \text{ or } k = 2 \\ -\frac{1}{4n+k} & \text{if } k = 3 \text{ or } k = 0 \end{cases}$$

Both series $\sum_{n\geq 1} a_n$ and $\sum_{n\geq 1} a_n(-1)^n$ converges. However, $\sum_{n\geq 1} |a_n|$ diverges (no absolute convergence).

Ex 4.

We assume that $f(x) = \sum_{n\geq 0} a_n x^n$ converges on $(-R, R)$.

a) We have to show the series $\sum_n \frac{a_n}{n+1} x^{n+1}$ converges on $(-R, R)$. We have:

$$\left|\frac{a_n}{n+1} x^{n+1}\right| = |a_n x^n| \left|\frac{x}{n+1}\right| \leq |a_n x^n| R.$$

Since $|x| < R$, the series f converge absolutely at x such:

$$\sum_{n\geq 0} \left|\frac{a_n}{n+1} x^{n+1}\right| \leq \sum_{n\geq 0} |a_n x^n| R < +\infty.$$

Thus, the series converges on $(-R, R)$.

We deduce that the convergence is uniformly on $[-\tilde{R}, \tilde{R}]$ with $\tilde{R} < R$. In particular, we can exchange the summation and differential operator: for any $|x| < R$

$$\left(\sum_{n\geq 0} \frac{a_n}{n+1} x^{n+1}\right)' = \sum_{n\geq 0} \frac{a_n}{n+1}(x^{n+1})' = \sum_{n\geq 0} a_n x^n = f(x).$$

b) We can add a constant term "a_{-1}" to the power series:

$$\tilde{F}(x) = 17 + a_0 x + \frac{a_1}{2}x^2 + \ldots + \frac{a_n}{n+1}x^{n+1} + \ldots$$

Ex 7. [3pts]

a) We use the ratio test: the series $\sum_n a_n$ if $\lim_{n\to+\infty} |a_{n+1}/a_n| = r < 1$. Indeed, take $|x| < 1/L$, the series $\sum_{n\geq 0} a_n x^n$ converges since: [2pts]

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{a_{n+1}}{a_n}x\right| \stackrel{n\to+\infty}{\longrightarrow} L \cdot |x| < 1.$$

b) If $L = 0$, then one can perform the previous computation with any x. Thus, the series converges everywhere. [1pt]

Ex 8. [3pts]

a) Taking $x = 0$ leads to $a_0 = b_0$. We also know that the differential of the series is also well-define on $(-R, R)$ and that the convergence is uniform. Therefore: [1pt]

$$\sum_{n\geq 1} n a_n x^{n-1} = \sum_{n\geq 1} n b_n x^{n-1}.$$

Taking once again the value at $x = 0$ leads to $a_1 = b_1$. We can proceed iteratively taking as many derivative as needed to deduce that $a_n = b_n$ for any n. [1pt]

b) We must have (a_n) such that:

$$\sum_{n\geq 0} a_n x^n = \sum_{n\geq 1} n a_n x^{n-1}.$$

Writing $\tilde{n} = n - 1$, we obtain:

$$\sum_{n\geq 0} a_n x^n = \sum_{\tilde{n}\geq 0} (\tilde{n}+1) a_{\tilde{n}+1} x^{\tilde{n}}.$$

We deduce from *a*) that $a_n = (n+1)a_{n+1}$ in other words: [1pt]

$$a_{n+1} = \frac{1}{n+1} a_n.$$

Since $a_0 = 1$, we get $a_1 = 1$, $a_2 = 1/2 \cdot a_1 = 1/2$, $a_3 = 1/3 \cdot a_2 = 1/6$. We deduce iteratively that $a_n = 1/n!$.

2 Chapter 6.6

Ex 2.

a) We use that $\cos x = 1 - x^2/2 + x^4/4! + \ldots = \sum_{n\geq 0}(-1)^n \cdot x^{2n}/(2n)!$. Thus,

$$\begin{aligned} x\cos x^2 &= x\sum_{n\geq 0}(-1)^n(x^2)^{2n}/(2n)! = \sum_{n\geq 0}(-1)^n x^{4n+1}/(2n)! \\ &= x - x^5/2 + x^9/4! - \ldots \end{aligned}$$

The convergence is for any x (use ratio test).

b) We notice that:

$$\left(\frac{1}{1+4x^2}\right)' = \frac{-8x}{(1+4x^2)^2}.$$

Thus,

$$\frac{x}{(1+4x^2)^2} = -\frac{1}{8}\left(\frac{1}{1+4x^2}\right)'.$$

Moreover, for any $|4x| < 1$, we have:

$$\frac{1}{1+4x^2} = 1 - (4x^2) + (4x^2)^2 - (4x^2)^3 \ldots = \sum_{n\geq 0}(-1)^n 4^n x^{2n}.$$

Therefore,

$$\begin{aligned} \frac{x}{(1+4x^2)^2} &= -\frac{1}{8}\left(\frac{1}{1+4x^2}\right)' = -\frac{1}{8}\left(\sum_{n\geq 0}(-1)^n 4^n x^{2n}\right)' \\ &= -\frac{1}{8}\sum_{n\geq 1}(-1)^n 2n 4^n x^{2n-1}. \end{aligned}$$

The representation of the series is valid for $|x| < 1/4$.

c) We use that $\log(1+x) = \sum_{n\geq 1}(-1)^{n+1}/nx^n$ valid for $|x| < 1$. Thus,

$$\log(1+x^2) = \sum_{n\geq 1}(-1)^{n+1}/n(x^2)^n = \sum_{n\geq 1}(-1)^{n+1}/nx^{2n}.$$

The series representation is valid only for $|x| < 1$ (even though the function $\log(1+x^2)$ is defined for all $x \in \mathbb{R}$).

Ex 7.

a) Consider $g(x) = 1/(1+x^2)$. Its power series representation gives:

$$g(x) = \sum_{n\geq 0}(-1)^n x^{2n}$$

which only converges for $|x| < 1$ even though g is infinitely differentiable and well-defined on $\mathbb{R}$.

Remark. The 'explosion' of the power-series at $|x| = 1$ can be better understood looking at g as a *complex function*: $g(z) = 1/(1 + z^2)$ with $z \in \mathbb{C}$. We observe that g has a singularity at $z = \mathbf{i}$ since $1 + \mathbf{i}^2 = 1 + (-1) = 0$. Thus, the radius of convergence of the power series cannot exceed $|\mathbf{i}| = 1$.

b) Consider $h(x) = \sin x + g(x)$ with the function g given by:

$$g(x) = \begin{cases} \exp(-1/x^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

The function h has the same power series representation as sin at $x = 0$ (all the derivative of g are zero at $x = 0$). However, since $g(x) \neq 0$ for $x \neq 0$ we always have $h(x) \neq \sin(x)$ for $x \neq 0$.

c) Consider the function:

$$f(x) = \begin{cases} \exp(-1/x) & \text{if } x > 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

The Taylor series of f at $x = 0$ is $a_n = 0$. Thus, the Taylor series agrees with f for $x \geq 0$ but is different for $x > 0$.

3 Chapter 7.2

Ex 2.

a) Take $f(x) = 1/x$. Since f is a decreasing function for $x > 0$, the min and max on an interval are reached at the right and left bound (respectively). Let $P = \{1, 3/2, 2, 4\}$, we have:

$$\begin{aligned} L(f,P) &= f(3/2)\cdot(3/2-1) + f(2)\cdot(2-3/2) + f(4)\cdot(4-2) = \frac{13}{12}, \\ U(f,P) &= f(1)\cdot(3/2-1) + f(3/2)\cdot(2-3/2) + f(2)\cdot(4-2) = \frac{11}{6}. \end{aligned}$$

Thus, $U(f,P) - L(f,P) = 9/12 = 3/4$

b) The error $U(f,P) - L(f,P)$ will be reduced (we get $1/2$)

c) Take for instance $\tilde{P} = \{1, 1.5, 2, 2.5, 3, 3.5, 4\} = \{1+k\cdot\Delta x, k = 0..8\}$ with $\Delta x = 1/2$. We obtain: $U(f,\tilde{P}) - L(f,\tilde{P}) = .375 < 2/5$.