MAT 371: Homework 6 (02/12)

1 Chapter 3.2

Ex 2. [4pts]

a) **Set** A: limit points are -1 and $+1$. Take the sequence $x_n = (-1)^{2n} + 2/(2n)$ in A converging to $+1$ and the sequence $y_n = (-1)^{2n+1} + 2/(2n+1)$ in A converging to -1. .5+.5pt

Set B: limit points are the interval $[0, 1]$ since Q is dense in $\mathbb{R}$.

b) **Set** A: the set is not open (it does not contain any neighborhood). It is also not closed since the limit point -1 is not contained in A. .5+.5pt

Set B: neither open (no neighborhood) or closed (limit points are not contained).

c) **Set** A: the point 0 is isolated in A. Actually, all the points are isolated except 1.

Set B: no point is isolated. For any x in B, we can find a sequence x_n in B that converges to x and different than x for all n. Take for instance $x_n = x + 1/n$ with n large enough such that $x_n \in B$. .5+.5pt

d) $\overline{A} = A \cup -1$. $\overline{B} = [0, 1]$. .5+.5pt

Ex 3.

a) Q is not open since it does not contain any neighborhood. It is not closed, the limit point $\sqrt{2}$ does not belong to $\mathbb{Q}$.

b) $\mathbb{N}$ is not open (no neighborhood). It is closed.

c) $\mathbb{R}$ $\{0\}$ is open: take $x \neq 0$. Let $\varepsilon = |x|/2$, then $V_\varepsilon(x) \subset \mathbb{R}$ $\{0\}$. It is not closed, since the limit point 0 does not belong to the set.

d) The set is not open since it doesn't contain any neighborhood. It is not closed either since the limit point $\sum_{n=1}^{+\infty} 1/n^2$ (i.e. the series converges) is not contained in the set.

e) The set is still no open but it is now closed. Indeed, since the series diverges there is no more limit points in the set.

Ex 6.

a) False: take $A = (-\infty, \pi) \cup (\pi, +\infty)$. A is open and contain all rational numbers. However it is different from $\mathbb{R}$.

b) False: consider $I_n = [n, +\infty)$. We have $\cap_{n\geq 0} I_n = \emptyset$.

c) True: suppose $\mathcal{O}$ open set. Take $x \in \mathcal{O}$ and consider $\varepsilon > 0$ such that $V_\varepsilon(x) \subset \mathcal{O}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $z \in \mathbb{Q}$ such that $|x - z| < \varepsilon$. We deduce that $z \in V_\varepsilon(x)$ and therefore $z \in \mathcal{O}$.

d) False: consider $A = \{\pi + 1/n \text{ for } n \geq 1\} \cup \{\pi\}$. The set is closed, infinite, and bounded. However, it does not contain any rational numbers.

e) True: the Cantor is an (infinite) intersection of closed sets. It is therefore closed.

Ex 8. [3pts]

a) The set $\overline{A \cup B}$ is always closed. .5pt

b) The set $A \backslash B = A \cap (B^c)$ is always open since A and B^c are both open. .5pt

c) The set $A^c \cup B$ is closed as (finite) union between two closed set. The complement $(A^c \cup B)^c$ is open. .5pt

d) The union gives simply the set B, thus closed. .5pt

e) Since A^c closed, we have $\overline{A^c} = A^c$. Moreover, since $\overline{A} \supset A$, we have $\overline{A}^c \subset A^c$. Therefore: 1pt

$$\overline{A}^c \cap \overline{A^c} = \overline{A}^c \cap A^c = \overline{A}^c$$

and we deduce that the set is open.

Ex 11.

a) We know that $A \cup B \subset \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B}$ is a closed set. Since $\overline{A \cup B}$ is the *smallest* closed set containing $A \cup B$, we deduce: $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

Since $A \subset A \cup B$, we have $\overline{A} \subset \overline{A \cup B}$. Similarly, $\overline{B} \subset \overline{A \cup B}$. Therefore, $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

b) This result does not extend to infinite union. For instance, the closure of $\mathbb{Q}$ is $\mathbb{R}$, but the union of the closure of each element of $\mathbb{Q}$ is still only $\mathbb{Q}$.

Ex 14.

a) Suppose E closed. Then the smallest closed set containing E is necessarily E. Thus, E closed implies $\overline{E} = E$.
Now suppose $\overline{E} = E$. Since $\overline{E}$ is by definition closed, we also have E closed.
Similarly, suppose E open. The largest open set contained in E is necessarily E. Thus, E open implies $E^o = E$.
Now suppose $E^o = E$. Since E^o is by definition open, we also have E open.

b) The set $\overline{E}^c$ is open and contained in E^c (since $\overline{E} \supset E$ implies $\overline{E}^c \subset E^c$). Therefore, $\overline{E}^c \subset (E^c)^o$.

On the other hand, suppose now $\mathcal{O}$ open set *in between* $\overline{E}^c$ and E^c:

$$\overline{E}^c \subset \mathcal{O} \subset E^c.$$

We have to show that $\mathcal{O} = \overline{E}^c$. Taking the complement, we find:

$$\overline{E} \supset \mathcal{O}^c \supset E.$$

Thus, $\mathcal{O}^c$ is a closed set containing E. Since $\overline{E}$ is the smallest closed set containing E, we have: $\mathcal{O}^c \supset \overline{E}$. We conclude that $\mathcal{O}^c = \overline{E}$ and therefore $\mathcal{O} = \overline{E}^c$. Thus, $\overline{E}^c$ is the largest open set contained in E^c.

We proceed similarly to prove that $(E^o)^c = \overline{E^c}$. First, from $E^o \subset E$ we deduce $(E^o)^c \supset E^c$ and therefore $(E^o)^c \supset \overline{E^c}$. Second, suppose there exists F such that $(E^o)^c \supset F \supset E^c$. Then $E^o \subset F^c \subset E$. Since E^o is the largest open set contained in E, we necessarily have $E^o = F^c$. Thus, $(E^o)^c$ is the smallest set containing E^c.

2 Chapter 3.3

Ex 2. [3pts]

a) $\mathbb{N}$ is not compact since it is not bounded. .5pt

b) $\mathbb{Q} \cap [0,1]$ is not closed, the limit points are all $[0,1]$. .5pt

c) The Cantor set is closed (intersection a closed set) and bounded (by $M = 1$). Thus, it is compact. .5pt

d) $\{1 + \cdots + 1/n^2 \ : \ n \in \mathbb{N}\}$: not compact since it does not contain the limit point $x = \sum_{n=1}^{\infty} 1/n^2$. .5pt

e) $\{1, 1/2, 2/3 \ldots, n/(n+1)\}$: the only limit point 1 is contained in the set. Since the set is also bounded, the set is compact. 1pt