MAT 371: Homework 7 (03/12)

1 Chapter 3.3

Ex 4. [4pts]

a) $K \cap F$ is compact: it is closed as intersection of two closed sets and bounded since it is contained in a bounded set $K \cap F \subset K$. [1pt]

b) Not necessarily compact since it might not be bounded. Take $F = K = [0, 1]$. We obtain $\overline{F^c \cup K^c} = (-\infty, 0] \cup [1, +\infty)$ unbounded therefore not compact. [1pt]

c) Not compact: take $K = [0, 2]$ and $F = [0, 1]$. Then $K \backslash F = (1, 2]$ not closed and therefore not compact. [1pt]

d) Compact: it is closed and contained in a bounded set $\overline{K \cap F^c} \subset \overline{K} = K$ bounded. [1pt]

Ex 8.

a) Since d is an infinimum, there exists a sequence $(x_n, y_n)_n$ with $x_n \in K$ and $y_n \in L$ such that: $|x_n - y_n| \to d$. Since K and L are compact, we can extract converging subsequence: there exists n_k and $x \in K$, $y \in L$ such that $x_{n_k} \to x$ and $y_{n_k} \to y$. Thus, $|x_{n_k} - y_{n_k}| \to |x - y|$. Therefore $d = |x - y|$. If K and L are disjoint, then $x \neq y$, therefore $d = |x - y| > 0$.

b) If the sets K and L are only closed but not compact, d might be equal to zero even if K and L are disjoint. They might intersect "at infinity". Take for example

$$K = \mathbb{N} \quad \text{and} \quad F = \{n + \frac{1}{n} \text{ for } n \in \mathbb{N}\}.$$

We have $K \cap F = \varnothing$, but taking $x_n = n$ and $y_n = n + 1/n$, we obtain:

$$d \leq \lim_{n \to +\infty} |y_n - x_n| = 0.$$

Thus, $d = 0$.

2 Chapter 3.4

Ex 1. [2pts]

Since P and K closed, we deduce $P \cap K$ closed as well. It is also bounded since $P \cap K \subset K$ bounded. Therefore, $P \cap K$ compact. 1pt

The intersection is however not necessarily perfect: take $P = [0, 1]$ perfect and $K = \{0\} \cup \{1\}$ compact. We have $P \cap K = K$ not perfect (0 and 1 are isolated). 1pt

Ex 7.

a) Take $x, y \in \mathbb{Q}$ with $x \neq y$. Consider a non-rational number c between x and y (take for example $c = \frac{x+y}{2} + \frac{|y-x|}{2}\frac{\pi}{4}$). Let $A = (-\infty, c) \cap \mathbb{Q}$ and $B = (c, +\infty) \cap \mathbb{Q}$. We have A and B separated with $A \cup B = \mathbb{Q}$ and $x \in A$, $y \in B$. Thus, $\mathbb{Q}$ is totally disconnected.

b) The set of irrational numbers is also totally disconnected. For x, y distinct non-rational number, we can find (by density) c rational in between x and y. We can then use the same argument as in a).

3 Chapter 3.5

Ex 5.

Suppose $\mathbb{R} = \cup_{n=1}^{+\infty} F_n$. Taking the complement gives $\varnothing = \cap_{n=1}^{+\infty} O_n$ with $O_n = F_n^c$ open set. Since F_n does not contain any interval, we would like to deduce that O_n is dense. Indeed, take any two points $a < b$. Since (a, b) is not contained in F_n, there exists $a < x < b$ such that $x \notin F_n$. Therefore $x \in O_n$ and we deduce that O_n is dense in $\mathbb{R}$. We can now use Baire's theorem to conclude that $\cap_{n=1}^{+\infty} O_n$ cannot be empty.

Ex 9.

a) $\overline{A} = [0, 5]$. Thus A is not dense or no-where dense.

b) $\overline{B} = 0 \cup 1/n$ with $n \in \mathbb{N}$. Thus B is no-where dense.

c) The set of irrationals is dense in $\mathbb{R}$.

d) The Cantor set C is no-where dense: $\overline{C} = C$ and C is of length zero, therefore does not contain any (nonempty) open intervals.

4 Chapter 4.2

Ex 2. [4pts]

a) Denote $f(x) = 5x - 6$ and $L = 9$. Fix $\varepsilon = 1$, then take $\delta = \varepsilon/5$. If $|x - 3| < \delta$, then: 1pt

$$|f(x) - L| = |5x - 15| = 5|x - 3| < 5\delta = \varepsilon.$$

b) Let $f(x) = \sqrt{x}$ and $L = 2$. Fix $\varepsilon = 1$, then take $\delta = 2\varepsilon$. If $|x-4| < \delta$, then $\sqrt{x} > 0$ therefore $|\sqrt{x} + 2| > 2$. We deduce that **1pt**

$$|f(x) - L| = |\sqrt{x} - 2| < |\sqrt{x} - 2| \cdot \frac{|\sqrt{x}+2|}{2} = \frac{|x-4|}{2} < \frac{\delta}{2} = \varepsilon.$$

c) Fix $\varepsilon = 1$. Take $\delta = \pi - 3 = .141592...$. Then for $|x - \pi| < \delta$, we have: **1pt**

$$x > 3 \quad \text{and} \quad x < 4.$$

Thus, $[[x]] = 3$. Therefore: $|[[x]] - 3| = 0 < \varepsilon$.

d) Same as c). **1pt**