MAT 371: Homework 8 (03/21)

1 Quiz review

Sequence.

a) For any $\varepsilon > 0$, there exists $N > 0$ such that:

$$|a_m - a_n| < \varepsilon \qquad \text{for any } m \geq n \geq N. \tag{1}$$

b) We have to find M such that $|a_n| \leq M$ for any n. We are going to use the Cauchy criteria to show that (a_n) is bounded 'at infinity' (i.e. for large values of n).

Take $\varepsilon = 1$. Denote N such that the equation (2) is satisfied. In particular, for $n = N$, we find:

$$|a_m - a_N| < 1 \quad \Rightarrow \quad |a_m| < |a_N| + 1 \quad \text{for all } m \geq N.$$

Therefore, (a_n) stays below $|a_N| + 1$ for large n. To conclude, we denote $M = \max(|a_0|, |a_1|, \ldots, |a_N|, |a_N| + 1)$, we deduce that $|a_n| \geq M$ for any n. Therefore, (a_n) bounded.

c) From b), we know that the sequence (a_n) is bounded. Let's denote M an upper-bound. Fix $\varepsilon > 0$. Since (a_n) Cauchy, we apply the Cauchy criteria with $\tilde{\varepsilon} = \varepsilon/2M$: there exists $\tilde{N}$ s.t.

$$|a_m - a_n| < \varepsilon/2M \qquad \text{for any } m \geq n \geq \tilde{N}. \tag{2}$$

We deduce that:

$$\begin{aligned} |a_m^2 - a_n^2| &= |a_n - a_m| \cdot |a_n + a_m| \leq |a_n - a_m| \cdot (|a_n| + |a_m|) \\ &\leq 2M|a_n - a_m| < 2M \cdot \varepsilon/2M = \varepsilon \quad \text{for any } m \geq n \geq \tilde{N}. \end{aligned}$$

We conclude that (a_n^2) is also Cauchy.

Series.

a) For any $\varepsilon > 0$, there exists $N > 0$ such that:

$$|a_m + a_{m-1} + \cdots + a_{n+2} + a_{n+1}| < \varepsilon \qquad \text{for any } m \geq n \geq N. \tag{3}$$

b) Since the series is Cauchy, we have $a_n \to 0$ in particular (a_n) bounded by some M. Therefore:

$$\begin{aligned} |a_m^2 + a_{m-1}^2 + \cdots + a_{n+2}^2 + a_{n+1}^2| &\leq M \cdot |a_m| + M \cdot |a_{m-1}| + \cdots + M \cdot |a_{n+2}| + M \cdot |a_{n+1}| \\ &= M \cdot (a_m + a_{m-1} + \cdots + a_{n+2} + a_{n+1}) \qquad (a_n \geq 0) \\ &= M \cdot (|a_m + a_{m-1} + \cdots + a_{n+2} + a_{n+1}|) < M \cdot \varepsilon. \end{aligned}$$

We can conclude using the same methodology as before (see **Sequence** problem).

c) No. We can find a counter-example: $a_n = (-1)^n/\sqrt{n}$. The series $\sum a_n$ converges (alternative series criteria), but the series with $a_n^2 = 1/n$ does not converge.

2 Chapter 4.2

Ex 5. [4pts]

a) Fix $\varepsilon > 0$. Let $\delta = \varepsilon/3$, for any $|x - 0| < \delta$, we have: [1pt]

$$|f(x) - f(0)| = |3x + 4 - 4| = |3x| = 3|x| < 3\delta = \varepsilon.$$

Thus, f continuous at $x = 0$.

b) Fix $\varepsilon > 0$. Take $\delta = \min(1, \varepsilon)$. For any $|x| < \delta$, we have $|x| < 1$ therefore $|x^2| < 1$. We deduce: [1pt]

$$|x^3 - 0^3| = |x^2| \cdot |x| \leq |x| < \varepsilon.$$

since $|x| < \delta$ induces $|x| < \varepsilon$.

c) Fix $\varepsilon > 0$. Denote $\delta = \min(\varepsilon/6, 1)$. For any $|x - 2| < \delta$, we have $|x| < 3$ and therefore $|x + 3| < 6$. We deduce: [1pt]

$$|f(x) - f(2)| = |x^2 + x - 1 - 5| = |x^2 + x - 6| = |(x + 3)(x - 2)| < 6 \cdot |x - 2| < \varepsilon.$$

d) Fix $\varepsilon > 0$. Take $\delta = \min(1, \varepsilon)$. For any $|x - 3| < \delta$, we have $|x| > 2$. Thus, $1/|x| < 1/2$ [1pt]

$$|f(x) - 1/3| = \left|\frac{3 - x}{3x}\right| = \left|\frac{1}{3x}\right| \cdot |3 - x| < \frac{1}{6}\delta < \varepsilon.$$

Ex 9. [2pts]

a) Fix $M > 0$. There exists $\delta > 0$, such that $1/\delta > \max(M, 1)$. Thus, for any $0 < x < \delta$, we have: [1pt]

$$\frac{1}{x} > \frac{1}{\delta} > M.$$

Since $\frac{1}{x} > 1$, we deduce $\frac{1}{x^2} > M$ for any $0 < |x - 0| < \delta$.

b) For any $\varepsilon > 0$, there exists $M > 0$ such that: [1pt]

$$|f(x) - L| < \varepsilon \qquad \text{for any } x > M.$$

Application: fix $\varepsilon > 0$. Consider $M > 1/\varepsilon$. For any $x > M$, we have:

$$|1/x - 0| = |1/x| < \varepsilon.$$

Therefore $1/x \to 0$ as $x \to +\infty$.

3 Chapter 4.3

Ex 6. [4pts]

a) Yes: take f the Heaviside function (discontinuous at $x = 0$) 1pt

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Consider $g(x) = 1 - f(x)$ not continuous at $x = 0$ either. We have $f(x)g(x) = 0$ for all x. It is therefore a continuous function (a constant function is continuous). Similarly, $f(x) + g(x) = 1$ for all x, also continuous.

Remark. The key here is to notice that the two 'jumps' of f and g compensate and annihilate each other (i.e. f jumps by $+1$ whereas g jumps by -1).

b) Not possible: if f continuous at $x = 0$ and g not continuous at $x = 0$, then the sum $f + g$ is always discontinuous at $x = 0$. Indeed, the 'big jump' of g at $x = 0$ cannot be compensated by f which only does a 'small jump'. 1pt

c) Yes: take $f(x) = 0$. The product $f(x)g(x)$ is automatically zero and thus continuous. 1pt

d) Yes: take 1pt

$$f(x) = \begin{cases} 2 & \text{if } x \geq 0 \\ 1/2 & \text{if } x < 0. \end{cases}$$

The function f is not continuous at $x = 0$, but $f(x) + 1/f(x)$ is a constant function (equal to $2 + 1/2$) thus continuous.

e) Not possible: if $[f(x)]^3$ continuous at $x = 0$, then taking $g(x) = x^{1/3}$ (continuous) we have $g(f(x)^3)$ continuous as well. Thus, $g(f(x)^3) = f(x)$. Therefore we would have f continuous as well.

Ex 9.

The set K can be written as $K = h^{-1}(\{0\})$, i.e. it is the pre-image of the singleton $\{0\}$. Since $\{0\}$ is closed and h continuous, we deduce that $h^{-1}(\{0\})$ is closed as well (i.e. $h^{-1}(\text{"}closed\text{"})$ is closed).