MAT 371: Homework 9 (04/09)

1 Chapter 5.2

Ex 2. [4pts]

a) Possible: take

1pt

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 1. \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 1. \end{cases}$$

both functions are not continuous and therefore not differentiable at $x = 0$. However, $fg = 0$ for all x and therefore differentiable.

b) Possible: take f Heaviside and $g = 0$.

1pt

c) Not possible

1pt

d) Possible: take

$$h(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

1pt

Then consider the function $f(x) = x^2 h(x)$. The function is continuous and differentiable at $x = 0$ (with $f'(0) = 0$), but it is not continuous or differentiable anywhere else.

Ex 5.

a) $f_a(x)$ continuous at zero for any $0 \leq a$.

b) For $a > 0$, we have $f_a'(x) = (a-1)x^{a-1}$. Thus f_a differentiable at $x = 0$ if $a \geq 1$. In this case, f_a' is also continuous.
If $a = 0$, then $f_a(x) = 1$ and it is always differentiable.

c) f_a is twice-differentiable if either $a = 0$, $a = 1$ or $a \geq 2$.

Ex 6.

a) Since $x \to c$ is equivalent to $x - c \to 0$, we can denote $h = x - c$ and write down:

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = \lim_{h \to 0} \frac{g(c+h) - g(c)}{h}.$$

b) We can also write by taking $h = c - x$:

$$g'(c) = \lim_{h\to 0} \frac{g(c-h) - g(c)}{-h}.$$

Therefore, by taking the average of the two previous equations:

$$g'(c) = \frac{1}{2}\left(\lim_{h\to 0} \frac{g(c+h) - g(c)}{h} \quad + \quad \lim_{h\to 0} \frac{g(c-h) - g(c)}{-h}\right) = \lim_{h\to 0} \frac{g(c+h) - g(c-h)}{2h}.$$

2 Chapter 5.3

Ex 1.

a) If f' continuous on $[a, b]$ (compact), then we must have f' bounded on $[a, b]$. Thus, there exists M such that:

$$|f'(x)| \leq M \quad \text{for any } x \in [a, b].$$

Moreover, applying the Mean-Value-Theorem, we deduce that for any $x, y \in A$:

$$|f(x) - f(y)| = |f'(c)||x - y|,$$

with $c \in (a, b)$. Since $|f'(c)| \leq M$, we have:

$$|f(x) - f(y)| \leq M|x - y|.$$

We conclude that f is Lipschitz on $[a, b]$.

b) Consider $M = \sup_{x\in[a,b]} |f'(x)|$. If we know that $|f'(x)| < 1$ on $[a, b]$, we deduce that $M < 1$ since $[a, b]$ is compact. Therefore, the function f is contractive on $[a, b]$.

Remark. Notice that the argument does not work on $\mathbb{R}$. For instance consider $f(x) = \sqrt{1 + x^2}$. We have $|f'(x)| < 1$ for all $x \in \mathbb{R}$, but $\sup_{x\in\mathbb{R}} |f'(x)| = 1$ and f is not a contraction on $\mathbb{R}$.

Ex 2. [3pts]

Let's proceed by contradiction: suppose there exist x, y with $x \neq y$ such that $f(x) = f(y)$. Then, by Rolle's theorem, there exists $c \in A$ such that $f'(c) = 0$ contradiction. Therefore, the function f has to be injective (i.e. one-to-one). [2pts]

However, if we consider $f(x) = x^3$ on the interval $I = [-1, 1]$. It is clear that f is one-to-one (there exists an inverse) even though $f'(0) = 0$. Therefore $f' = 0$ is *sufficient* condition to be one-to-one, it is not however a *necessarily* condition. [1pt]

Ex 6. [3pts]

a) It is a direct application of the MVT: 1pt

$$|g(x)| = |g(x) - g(0)| = |g'(c)(x - 0)| \leq Mx,$$

where $c \in (0, x)$.

b) Applying the previous result to h', we deduce that: $h'(x) \leq Mx$. 1pt
Denote $\varphi(x) = Mx^2/2 - h(x)$. It satisfies $\varphi'(x) = Mx - h'(x) \geq 0$ (increasing function) and $\varphi(0) = 0$. Thus, $\varphi(x) \geq 0$ for all $x \in [0, a]$ and therefore $Mx^2/2 \geq h(x)$. By a similar argument with $Mx^2/2 + h(x)$ we deduce that $Mx^2/2 \geq -h(x)$ and therefore $Mx^2/2 \geq |h(x)|$.

c) More generally, we will have $|f(x)| \leq Mx^3/6$ if $f(0) = f'(0) = f''(0) = 0$. 1pt