MAT 371: Midterm 2 Mar. 28, 2019 Duration: 1:15

Exercise 1. 25pts

- No: $O = (0,2)$, $F = [-1,1]$. Then $O \cap F = (0,1]$ not open or closed. 3+3pts
- No: $F_n = [-1+1/n, +1-1/n]$. We have $\cup_n F_n = (-1,1)$ not closed. 4+3pts
- Yes: $K_1 \cup K_2$ is closed (finite union of closed set) and bounded. 3+3pts
- No: $C = (0,4)$, $O = (0,1) \cup (3,5)$. Then $C \cap O = (0,1) \cup (3,4)$ no connected. 3+3pts

Exercise 2. 20pts

a) $f(A) = [0,4]$, $f^{-1}(f(A)) = [-2,2]$. 3+3pts
$f^{-1}(B) = (-1,1)$, $f(f^{-1}(B)) = [0,1)$. 3+5pts

b) Take $x \in A$ and $y = f(x)$. We have by definition $y \in f(A)$ (y is the image of x by f). Thus, we can also see x as a pre-image of y, therefore $x \in f^{-1}(f(A))$. We deduce $A \subset f^{-1}(f(A))$. 6pts

Extra) The key is to take a function f **not** continuous. Take for instance f the Heaviside function and $O = (-1/2, 3/2)$. We have $f^{-1}(O) = [0, +\infty)$ not open. 5pts

Exercise 3. 30pts

a) f is continuous on I as the inverse of the continuous function $x \to x$ which is not zero on I. 6pts

However, f is not uniformly continuous on I. Indeed, take $x = 1/n$ and $y = 2/n$. We have $|x-y| \to 0$ as $n \to +\infty$, but: 6pts

$$|f(x) - f(y)| = |n/2| \to +\infty$$

as $n \to +\infty$. Thus, even though $|x-y|$ is small, the increment $|f(x) - f(y)|$ will become large.

Since f is continuous on $[.001, 1]$ compact, the function f is uniformly compact on $[.001, 1]$. 6pts

b) The function g is continuous on $\mathbb{R}^*$ (i.e. $\mathbb{R}$ without $\{0\}$) for any value of α. It only remains to check if g is continuous at $x = 0$. We have: 8pts

$$\begin{aligned} \lim_{x \to 0-} g(x) &= 3 \\ \lim_{x \to 0+} g(x) &= \frac{1}{\alpha}. \end{aligned}$$

Thus, we need to have $3 = 1/\alpha$ and thus $\alpha = 1/3$. 4pts

Extra. g uniformly continuous on $\mathbb{R}$. It is uniformly continuous on $\mathbb{R}^- = (-\infty, 0]$ since g is a linear. On $\mathbb{R}^+ = [0, +\infty)$, we notice that:

5pts

$$|g(x) - g(y)| = \left|\frac{1}{\alpha + x^2} - \frac{1}{\alpha + y^2}\right| = \left|\frac{y^2 - x^2}{(\alpha + x^2)(\alpha + y^2)}\right| = |x - y| \left|\frac{y + x}{(\alpha + x^2)(\alpha + y^2)}\right|$$

The function $\left|\frac{y+x}{(\alpha+x^2)(\alpha+y^2)}\right|$ is upper-bounded by some constant M. Thus, $|g(x) - g(y)| \leq M|x - y|$ and we can deduce that g is uniformly continuous.

Exercise 4. 25pts

a) Take $f(x) = x^2 + 10$. The equation $f(x) = x$ does not have a (real) solution on $\mathbb{R}$. 5pts

b)
- Since $a \leq f(x) \leq b$, we deduce that $a - x \leq f(x) - x \leq b - x$. In particular for $x = a$ and $x = b$: 3+3pts

$$0 \leq f(a) - a \leq b - a, \qquad a - b \leq f(b) - b \leq 0.$$

  Thus, $g(x) = f(x) - x$ is positive at $x = a$ and negative at $x = b$.
- Since f and the identity function are continuous, the function g is also continuous. By the IVT, we deduce that there exists $x_* \in [a, b]$ such $g(x_*) = 0$. 4pts 5pts
- Since $g(x_*) = 0$, we deduce $f(x_*) - x_* = 0$ meaning that $f(x_*) = x_*$. Therefore f has a fixed-point x_* on $[a, b]$. 5pts