MAT 371: Practice midterm 2

Exercise 1. [open/closed]

a) Compute the closure, interior and frontier of the following sets

i) $\bar{A} = [0, 1],\ \mathring{A} = (0, 1),\ \partial A = \{0\} \cup \{1\}.$

ii) $\bar{B} = \{\pi\} \cup [10, 12],\ \mathring{B} = (10, 12),\ \partial B = \{\pi\} \cup \{10\} \cup \{12\}.$

iii) $\bar{C} = \mathbb{N},\ \mathring{C} = \varnothing,\ \partial C = \mathbb{N}.$

iv) $\bar{D} = \{0\} \cup D,\ \mathring{D} = \varnothing,\ \partial D = \bar{D}.$

b) $\partial A = \bar{A} \cap (\mathring{A})^c$ is an intersection of two closed set, therefore it is a closed set.

c) $A = (0, 1)$ and $B = (1, 2)$. We have $\overline{A \cap B} = \varnothing$ but $\bar{A} \cap \bar{B} = \{1\}$.

d) Take $A = \mathbb{Q}$.

Exercise 2. [connectivity]

a) $\mathbb{Q} \subset O_1 \cup O_2$ with $O_1 = (-\infty, \pi)$ and $O_2 = (\pi, +\infty)$ disjoint open sets.

b) $A \cup B$ is not necessarily connected (take $A = (0, 1)$ and $B = (10, 12)$). $A \cap B$ is connected since an intersection of two intervals is still an interval.

c) Suppose C a connected set. Take $\bar{C} \subset O_1 \cup O_2$ with O_1 and O_2 two disjoint sets.

- Since $\bar{C} \subset O_1 \cup O_2$, we also have $C \subset O_1 \cup O_2$. Moreover C is connected, therefore $C \subset O_1$ or $C \subset O_2$.
- Taking the closure of the previous result leads to $\bar{C} \subset \bar{O}_1$ or $\bar{C} \subset \bar{O}_2$.
- Since O_1 and O_2 are disjoint, we have $O_1 \subset O_2^c$. Since O_2^c closed, we deduce $\bar{O}_1 \subset O_2^c$. Similarly, $\bar{O}_2 \subset O_1^c$.
- We deduce that $\bar{C} \subset \bar{O}_1$ implies $\bar{C} \subset O_2^c$. Therefore $\bar{C} \cap O_2 = \varnothing$. Similarly $\bar{C} \subset \bar{O}_1$ implies $\bar{C} \cap O_2 = \varnothing$.
  We deduce that that $\bar{C}$ either does not intersect O_1 or O_2. Since it is contained in $O_1 \cup O_2$, we have $\bar{C} \subset O_2$ or $\bar{C} \subset O_1$. Therefore $\bar{C}$ connected.

Exercise 3. [continuity and limit]

- Since $x \to 1 + x^2$ is continuous on $\mathbb{R}$ and never equal to zero, its inverse, given by f, is continuous on $\mathbb{R}$.

- For similar reasons g is also continuous on $\mathbb{R}\backslash\{0\}$. We just need to study the continuity of g at $x = 0$. We check the left and right limit:

$$\begin{aligned} \lim_{x \to 0^-} g(x) &= \lim_{x \to 0^-} -\frac{1}{1 - x} = -1 \\ \lim_{x \to 0^+} g(x) &= \lim_{x \to 0^+} \frac{1}{1 + x} = 1. \end{aligned}$$

  Since the two limits do not coincide, g is not continuous at $x = 0$.

- Proceeding similarly with h, we find that:

$$\lim_{x\to 0^-} h(x) = 1 = \lim_{x\to 0^+} h(x).$$

Therefore h continuous at $x = 0$. We conclude that h is continuous on $\mathbb{R}$.e

Exercise 4. [continuity and open/closed]

a) $f(A) = [0,4]$, $f^{-1}(B) = [-3,3]$.

b) Take $g(x) = 1/x$ and $F = [1,+\infty)$ closed. We have $g(F) = (0,1]$ not closed.

c) Denote $O = F^c$ the complement of F which is an open set. The key is to notice that:

$$f^{-1}(F) = (f^{-1}(O))^c.$$

Indeed,

$$\begin{aligned} x \in f^{-1}(F) &\Leftrightarrow f(x) \in F \\ &\Leftrightarrow f(x) \notin O \qquad \text{with } O = F^c \\ &\Leftrightarrow x \notin f^{-1}(O) \\ &\Leftrightarrow x \in (f^{-1}(O))^c. \end{aligned}$$

Since $f^{-1}(O)$ is open (f being continuous), we deduce that $(f^{-1}(O))^c$ closed and therefore $f^{-1}(F)$ closed as well.

Exercise 5. [continuity and compactness]

a) Solving $f(x) = x$ leads to $x^2 - 2x + 1 = 0$. We find a zero at $x_1 = 1$.

b) The equation $g(x) = x$ leads to $x^2 + 1 = x^2$ which does not have a solution. Therefore, g has no fixed point on I. Notice that I is not compact.

c) -

Exercise 6. [continuity and compactness]

Suppose $f : [0,1] \to \mathbb{R}$ continuous functions satisfying:

$$f(x) > 0 \qquad \text{for any } x \in [0,1].$$

a) Since f continuous and $I = [0,1]$ compact, there exists a minimum $x_m \in I$ of f on I: $f(x) \geq f(x_m)$ for any $x \in I$. Since $f(x) > 0$ on I, we deduce that $f(x_m) > 0$. Denote $m = f(x_m)/2$. We deduce that $f(x) > m/2$ for any $x \in I$.

b) It does not. Consider $f(x) = 1/(1+x^2)$. It satisfies $f(x) > 0$ for any $x \in \mathbb{R}$. But there is no $m > 0$ such that $f(x) > m$ for all x. Indeed $f(x) \overset{x\to+\infty}{\longrightarrow} 0$.

Exercise 7. [uniform continuity]

a) Fix $\varepsilon > 0$. Take $\delta = \varepsilon/3 > 0$. We have for any $|x - y| < \delta$:

$$|f(x) - f(y)| = |3x + 1 - (3y + 1)| = |3(x - y)| = 3|x - y| < 3\delta = \varepsilon.$$

Therefore f uniformly continuous.

b) Take $B = \mathbb{R}$ and $g(x) = x^2$. g is not uniformly continuous on $\mathbb{R}$, the increment $|g(x) - g(y)|$ increases as $x, y \to +\infty$.

c) The function h is continuous on $[-2, 2]$. Indeed, h continuous on $[-2, 2]\backslash\{0\}$ as composition of continuous function. Furthermore, at $x = 0$, we have:

$$\lim_{x\to 0^-} h(x) = -3 = \lim_{x\to 0^+} h(x).$$

Since h continuous on $[-2, 2]$ and that $[-2, 2]$ compact, we deduce that h is uniformly continuous on $[-2, 2]$.

Exercise 8. [IVT]

A function $f : A \to \mathbb{R}$ is strictly increasing if for any $x < y$ on A it satisfies:

$$f(x) < f(y).$$

a) $f(x) = x^2$ on $\mathbb{R}$ is not increasing. The heaviside function

$$h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

is increasing but not strictly increasing.

b) $f(x) = -x^2$ is strictly increasing on $(-\infty, 0]$.

c) $f : [a, b] \to \mathbb{R}$ continuous and strictly increasing. We denote $J = f([a, b])$.

- Since f continuous and $[a, b]$ compact, there exists a minimum and maximum of f denoted (respectively) $m = f(x_m)$ and $M = f(x_M)$ with $x_m, x_M \in [a, b]$.

  $$m \leq f(x) \leq M.$$

  Therefore $J \subset [m, M]$. Since $m = f(x_m)$ and $M = f(x_M)$, we also have $m, M \in J$. Since J is an interval (connected set), we deduce that $[m, M] \subset J$. Therefore $J = [m, M]$.
  Notice that since f increasing, we must have $m = f(a)$ and $M = f(b)$ (the min. is at the left and the max. is at the right).

- Take $y \in J$. We have $f(x_m) \leq y \leq f(x_M)$. By the IVT, there exists $x \in [x_m, x_M]$ such that $f(x) = y$.
  To show that such x is unique, suppose that there exists $\tilde{x}$ another point in $[a, b]$ such that $f(\tilde{x}) = y$. If $\tilde{x} \neq x$, then we have $\tilde{x} < x$ or $\tilde{x} > x$. Since f is strictly increasing, we deduce $f(\tilde{x}) < f(x)$ or $f(\tilde{x}) > f(x)$. Contradiction $f(\tilde{x}) = f(x) = y$.