MAT 342: Homework 3 (09/13)

1 Section 1.4

Ex. 1)

a) In general, $AB + BA \neq 2AB$.

B) In general, $AB - BA \neq 0$.

Ex. 3) 2pts

Take $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Ex. 7)

Since $A = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, we find:

$$A^2 = A^3 = A.$$

Thus, $A^n = A$ for any $n \geq 1$.

Ex. 9)

Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we find:

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A^4 = 0.$$

Thus, $A^n = 0$ for any $n \geq 4$.

Ex. 16) 2pts

The matrix A is non-singular, so A^{-1} exists. Now, we have to show that A^T is non-singular, meaning that the inverse of A^T exists. In other words, we have to show that there exists a matrix B such that:

$$B\,A^T = I,$$

where I is the identity matrix. Thus, the matrix B has also to satisfy that:

$$\left(B\,A^T\right)^T = I^T \;\Rightarrow\; (A^T)^T B^T = I^T$$
$$\Rightarrow\; AB^T = I.$$
1 pt

Therefore, B^T has to be the inverse of A: $B^T = A^{-1}$. Thus, $B = (A^{-1})^T$.

1 pt

We conclude that A^T is non-singular and its inverse is given by: $(A^{-1})^T$. Indeed:

$$(A^{-1})^T A^T = \left(AA^{-1}\right)^T = I^T = I.$$

2 Section 1.5

Ex. 7) 3pts

We apply the method of row-elimination to A:

- $(2) \to (2) - 3(1)$: $E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$. We obtain: $E_1 A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$.
- $(1) \to (1) - (2)$: $E_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. We obtain: $E_2 E_1 A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.
- $(1) \to \frac{1}{2}(1)$: $E_3 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$. We obtain: $E_3 E_2 E_1 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

Therefore, we have (**b)**):

$$A^{-1} = E_3 E_2 E_1.$$

1 pt

To write the matrix A as a product of elementary matrices, we notice[1]:

$$E_3 E_2 E_1 A = I \quad \Rightarrow \quad A = E_1^{-1} E_2^{-1} E_3^{-1},$$
1 pt

with:

$$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad E_3^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}.$$
1 pt

Ex. 8) 3pts

b) Let E be the row manipulation: $(2) \to (2) + (1)$): (i.e. $E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$)

$$EA = \begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix}.$$

Multiplying on the left by the inverse of E, we obtain that $A = LU$ with:

$$L = E^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix}.$$

d) Applying row elimination, we obtain:

$$E_3 E_2 E_1 A = \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$ $\boxed{\text{1 pt}}$

with:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \; E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

Thus, $A = LU$ with:

$$U = \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$ $\boxed{\text{1 pt}}$

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}.$$ $\boxed{\text{1 pt}}$

[1] Be careful with the order!