MAT 342: Homework 5 (10/04)

1 Section 3.3

Ex. 2) 2pts

a) Linear independent: $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1 \neq 0.$ $\boxed{.5 \text{ pt}}$

b) Linear dependent: 4 vectors in $\mathbb{R}^3$ (dimension 3) are necessarily dependent. $\boxed{.5 \text{ pt}}$

c) Linear dependent: $\begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ -2 & -2 & 0 \end{vmatrix} = 0.$ $\boxed{.5 \text{ pt}}$

d) Linear dependent (determinant is zero).

e) Linear independent: solving $c_1 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ leads to $c_1 = c_2 = 0$. $\boxed{.5 \text{ pt}}$

Ex. 6) 2pts

Let's test if the vectors $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$ are linearly independent. Take c_1, c_2 and c_3 such that:

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = \mathbf{0}.$$

Replacing the $\mathbf{y}_i$ by the $\mathbf{x}_i$ expressions lead to:

$$\begin{aligned} c_1(\mathbf{x}_1 + \mathbf{x}_2) + c_2(\mathbf{x}_2 + \mathbf{x}_3) + c_3(\mathbf{x}_3 + \mathbf{x}_1) &= \mathbf{0} \\ \Rightarrow (c_1 + c_3)\mathbf{x}_1 + (c_1 + c_2)\mathbf{x}_2 + (c_2 + c_3)\mathbf{x}_3 &= \mathbf{0}. \end{aligned}$$ $\boxed{1 \text{ pt}}$

Since the three vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent, we must have:

$$c_1 + c_3 = 0\ , \quad c_1 + c_2 = 0 \quad \text{and} \quad c_2 + c_3 = 0$$ $\boxed{1 \text{ pt}}$

This leads to $c_3 - c_2 = 0$ and therefore $c_3 = c_2 = c_1 = 0$. Since it is the only solution, we must have the three vectors $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$ linearly independent.

Ex. 16)

We write $A = [\mathbf{a}_1 \dots \mathbf{a}_n]$ with $\mathbf{a}_i$ column vector of size $\mathbb{R}^m$. By assumption, the vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ are linearly independent. We now investigate the null space of A. Take $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}$. Writing $\mathbf{x} = (x_1, \dots, x_n)^T$ we deduce:

$$A\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}.$$

Since the vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ are linearly independent, we must have $x_1 = \cdots = x_n = 0$. Therefore, $\mathbf{x} = 0$. Thus, $N(A) = \mathbf{0}$.

2 Section 3.4

Ex. 3)

a) The vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent ($\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2 \neq 0$) and since they are in $\mathbb{R}^2$ vector space of dimension 2, the two vectors form a basis of $\mathbb{R}^2$.

b) Three vectors cannot be linearly independent in a vector space of dimension 2.

c) $\dim(\mathrm{Span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)) = 2$.

Ex. 8) 3pts

a) The vectors $\mathbf{x}_1$, $\mathbf{x}_2$ cannot span $\mathbb{R}^3$, since $Span(\mathbf{x}_1, \mathbf{x}_2)$ is a vector space of dimension at most 2 and $\mathbb{R}^3$ has dimension 3. $\boxed{\text{1 pt}}$

b) We have a basis if the vector $\mathbf{x}_3 = (\alpha, \beta, \gamma)^T$ satisfies: $det(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \neq 0$. We also have $\boxed{\text{1 pt}}$

$$\begin{vmatrix} 1 & 3 & \alpha \\ 1 & -1 & \beta \\ 1 & 4 & \gamma \end{vmatrix} = \begin{vmatrix} 1 & 3 & \alpha \\ 0 & -4 & \beta - \alpha \\ 0 & 1 & \gamma - \alpha \end{vmatrix} = -4(\gamma - \alpha) - (\beta - \alpha) = 5\alpha - \beta - 4\gamma,$$

thus, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ form a basis if $5\alpha - \beta - 4\gamma \neq 0$.

c) Take $\mathbf{x}_3 = (0, 0, 1)^T$. $\boxed{\text{1 pt}}$

Ex. 10)

Take $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_5$. They are linearly independent:

$$det(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_5) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 2 & 4 & 0 \end{vmatrix} = -2 \neq 0.$$

Since $\mathbb{R}^3$ is a vector space of dimension 3, we have necessarily that $Span(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_5) = \mathbb{R}^3$. Thus, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_5$ forms a basis of $\mathbb{R}^3$.

3 Section 3.5

Ex. 4) 3pts

Let $\mathcal{B}$ the basis[1] of $\mathbb{R}^2$ defined by:

$$\mathcal{B} = \left\{ \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}.$$

Define the transition matrix $U = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ between the basis $\mathcal{B}$ and the standard basis. $\boxed{.5 \text{ pt}}$

We deduce[2]:

$$U^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}. \qquad \boxed{1 \text{ pt}}$$

U^{-1} is the transition matrix from the *old* standard basis to the *new* basis $\mathcal{B}$. Thus,

$$\begin{aligned} [\mathbf{x}]_{\mathcal{B}} &= U^{-1}\mathbf{x} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} & \boxed{.5 \text{ pt}} \\ [\mathbf{y}]_{\mathcal{B}} &= \begin{pmatrix} 5 \\ -8 \end{pmatrix} & \boxed{.5 \text{ pt}} \\ [\mathbf{z}]_{\mathcal{B}} &= \begin{pmatrix} -1 \\ 5 \end{pmatrix}. & \boxed{.5 \text{ pt}} \end{aligned}$$

Ex. 5)

a) Let $U = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$. The matrix U is non-singular since $\det(U) = 1$ and thus the vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ form a basis of $\mathbb{R}^3$. U is a transition matrix from the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

The inverse matrix U^{-1} is the transition matrix from $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. It is given by:

$$U^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

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[1] It is a basis since the 2 vectors are linearly independent and in a vector space of dimension 2.

[2] If $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ non-singular, then: $U^{-1} = \frac{1}{det(U)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

b) Denote by $\mathcal{B}$ the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. We deduce that:

$$
\begin{aligned}
\left[\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}\right]_{\mathcal{B}} &= U^{-1}\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}, \\
\left[\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right]_{\mathcal{B}} &= \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \\
\left[\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}\right]_{\mathcal{B}} &= \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.
\end{aligned}
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