MAT 342: Homework 6 (10/11)

1 Section 3.5

Ex. 7) 3pts

To transform from the basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$, we first transform from $\{\mathbf{w}_1, \mathbf{w}_2\}$ to $\{\mathbf{e}_1, \mathbf{e}_2\}$ (i.e. W), then from $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$ (i.e. V^{-1}). Thus,

$$S = V^{-1}W. \qquad \boxed{\text{1pt}}$$

where V and W are the transition matrices from resp. $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{w}_1, \mathbf{w}_2\}$ to the canonical basis. Thus,

$$W = VS = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 9 & 4 \end{bmatrix}. \qquad \boxed{\text{1pt}}$$

Therefore, $\mathbf{w}_1 = (5, 9)^T$, $\mathbf{w}_2 = (1, 4)^T$. $\boxed{\text{1pt}}$

2 Section 3.6

Ex. 1) 3pts

a) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$. Solving $A\mathbf{x} = \mathbf{0}$, we find that $\mathbf{x} = (-2\alpha, 0, \alpha)$. Thus, the nullspace of A is given by $N(A) = \mathrm{Span}\Big((-2, 0, 1)^T\Big)$ and $\dim(N(A)) = 1$. $\boxed{\text{.5 pt}}$

We deduce that the row and column space are subspaces of dimension 2 (i.e. $3 - 1$). Thus, we only have to take two linearly independent vectors to form a basis of those subspace. For instance,

$$\begin{aligned} \mathrm{Im}(A) &= \mathrm{Span}\Big((1, 2, 4)^T, (3, 1, 7)^T\Big) & \boxed{\text{.5 pt}} \\ \mathrm{Im}(A^T) &= \mathrm{Span}\Big((1, 3, 2)^T, (2, 1, 4)^T\Big). & \boxed{\text{.5 pt}} \end{aligned}$$

where $\mathrm{Im}(A)$ denotes the column space and $\mathrm{Im}(A^T)$ the row space.

b) Let $B = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$. Solving $B\mathbf{x} = \mathbf{0}$ yields:

$\boxed{\text{.5 pt}}$

$$N(B) = \left\{ \begin{pmatrix} 10/7\,\alpha \\ 2/7\,\alpha \\ 0 \\ \alpha \end{pmatrix} \quad \text{with } \alpha \text{ scalar} \right\} = \mathrm{Span}\Big((10, 2, 0, 7)^T\Big).$$

Thus, $\dim(N(B)) = 1$. We deduce that the row space and column space have dimension 3.

Since the column space is of dimension 3 in $\mathbb{R}^3$, we have: $\mathrm{Im}(A) = \mathbb{R}^3$. Thus, we can take for basis of $\mathrm{Im}(A)$ the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. For the row space, we already know that $\boxed{.5 \text{ pt}}$

$$\mathrm{Im}(B^T) = \mathrm{Span}\Big((-3, 1, 3, 4)^T,\ (1, 2, -1, -2)^T,\ (-3, 8, 4, 2)^T\Big). \qquad \boxed{.5 \text{ pt}}$$

Since the dimension of the row space is 3, we have necessarily that those 3 vectors are linearly independent. Thus, they form a basis of the row space.

c) Let $C = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$. Solving $C\mathbf{x} = \mathbf{0}$ leads to:

$$N(C) = \mathrm{Span}\Big((10 - \frac{9 \cdot 7}{5}, \frac{3 \cdot 7}{5}, 3, -4)\Big).$$

Similarly to **b)**, we deduce that the standard basis of $\mathbb{R}^3$ is a basis of $\mathrm{Im}(A)$ and the 3 row vectors of A form a basis of $\mathrm{Im}(A^T)$.

Ex. 8)

Let A a $m \times n$ matrix with $m > n$ with $N(A) = \{\mathbf{0}\}$.

a) By the rank-nullity theorem, $\mathrm{Im}(A)$ is a vector space of dimension n. The column vectors are linearly independent but they do not span $\mathbb{R}^m$: $\dim(\mathrm{Im}(A)) = n < m$.

b) The problem $A\mathbf{x} = \mathbf{b}$ has no solution if $\mathbf{b}$ is not in the column space. If $\mathbf{b}$ is in the column space, then there exists a solution $\mathbf{x}$ and this solution is necessarily unique since $N(A) = \{\mathbf{0}\}$.

Proof of the last statement. Suppose there exists a second solution $\mathbf{z}$ such that $A\mathbf{z} = \mathbf{b}$. Then we have:

$$A\Big(\mathbf{x} - \mathbf{z}\Big) = A\mathbf{x} - A\mathbf{z} = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Thus, $\mathbf{x} - \mathbf{z}$ is in the null space of A. Since $N(A) = \{\mathbf{0}\}$, we have $\mathbf{x} - \mathbf{z} = \mathbf{0}$. Therefore, $\mathbf{x} = \mathbf{z}$ and we conclude that the solution is unique.

3 Section 4.1

Ex. 2) Using polar coordinates (i.e. $x_1 = r\cos\theta$, $x_2 = r\sin\theta$), we find:

$$L(\mathbf{x}) = (r\cos(\theta + \alpha),\ r\sin(\theta + \alpha))^T.$$

Thus, $L(r,\theta) = (r, \theta + \alpha)$. L is a rotation of angle α.

Ex. 3) Let L be a translation. L is not a linear application since:

$$L(2\mathbf{x}) = 2\mathbf{x} + \mathbf{a} \neq 2(\mathbf{x} + \mathbf{a}) = 2L(\mathbf{x}).$$

Ex. 6)

a) Not linear: $L(\mathbf{0}) \neq \mathbf{0}$.

b) Linear.

c) Linear.

d) Not linear: $L((3,3,3)) \neq 3L((1,1,1))$.

4 Section 4.2

Ex. 5) 4pts

There are two different methods to find the matrix representation A of L:

i) Geometric method: find the images of the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ through L (see figure 1). Then, write:

$$A = \begin{bmatrix} | & | \\ L(\mathbf{e}_1) & L(\mathbf{e}_2) \\ | & | \end{bmatrix}.$$

ii) Algebraic method: decompose A in term of rotation and symmetry. For example,

$$A = R\,S$$

where R is a rotation and S a symmetry.

a) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}.$ $\boxed{1 \text{ pt}}$

b) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$ $\boxed{1 \text{ pt}}$

c) $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}.$ $\boxed{1 \text{ pt}}$

d) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$ $\boxed{1 \text{ pt}}$

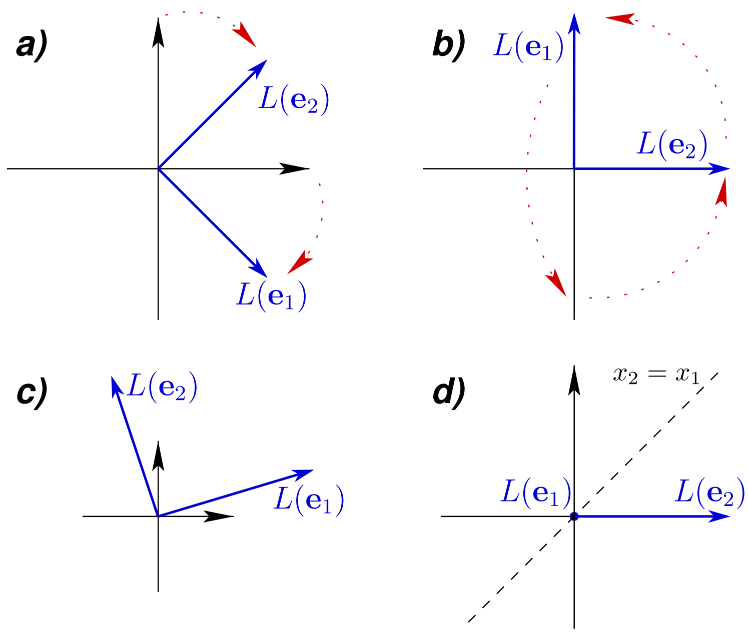

Figure 1: To obtain the matrix representation of a linear transformation L, one can simply find the image of the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$.