MAT 342: Homework 7 (10/18)

1 Section 4.2

Ex. 7) 3pts

We denote $\mathcal{B} = \{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$. We notice that $\mathcal{B}$ is a basis: $\det(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) = -1 \neq 0$.

a) $\mathbf{e}_1 = \mathbf{y}_3$, thus $[\mathbf{e}_1]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. $\boxed{.5 \text{ pt}}$

$\mathbf{e}_2 = \mathbf{y}_2 - \mathbf{y}_3$, thus $[\mathbf{e}_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. $\boxed{.5 \text{ pt}}$

$\mathbf{e}_3 = \mathbf{y}_1 - \mathbf{y}_2$, thus $[\mathbf{e}_3]_{\mathcal{B}} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. $\boxed{.5 \text{ pt}}$

b) Denote by U the transition matrix from the new basis $\mathcal{B}$ to the old one:

$$U = [\mathbf{y}_1\ \mathbf{y}_2\ \mathbf{y}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \qquad \boxed{.5 \text{ pt}}$$

Since $\mathcal{B}$ is a basis, the matrix U has to be non-singular (i.e. invertible).
For any vector $\mathbf{x}$ written in the canonical basis, we have: $[\mathbf{x}]_{\mathcal{B}} = U^{-1}\mathbf{x}$, where:

$$U^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}. \qquad \boxed{1 \text{ pt}}$$

Remark. Notice that: $[\mathbf{e}_1]_{\mathcal{B}} = U^{-1}(1,0,0)^T$ and similarly for $\mathbf{e}_2$ and $\mathbf{e}_3$.

Ex. 8) 3pts

a) In the basis $\mathcal{B}$, we have:

$$L(c_1, c_2, c_3) = \begin{pmatrix} c_1 + c_2 + c_3 \\ 2c_1 + c_3 \\ -2c_2 - c_3 \end{pmatrix}.$$

Thus, in the basis $\mathcal{B}$, the linear transformation L is represented by the matrix:

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix}.$$ $\boxed{1 \text{ pt}}$

b) i) $[\mathbf{x}]_{\mathcal{B}} = U^{-1}\mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$. Thus, $\mathbf{x} = 2\mathbf{y}_1 + 3\mathbf{y}_2 + 2\mathbf{y}_3$. We deduce that: $\boxed{1 \text{ pt}}$

$$[L(\mathbf{x})]_{\mathcal{B}} = D\,[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ -8 \end{pmatrix}.$$ $\boxed{1 \text{ pt}}$

Thus, $L(\mathbf{x}) = 7\mathbf{y}_1 + 6\mathbf{y}_2 - 8\mathbf{y}_3$. Therefore, in the canonical basis, we can write:

$$L(\mathbf{x}) = \begin{pmatrix} 5 \\ 13 \\ 7 \end{pmatrix}.$$

Remark. Another method consists in finding A the matrix representation of the linear transformation L in the canonical basis. For that, we use the formula (see figure 1):

$$A = U\,D\,U^{-1}.$$

where U and U^{-1} are the transition matrices from resp. the new basis to the canonical basis and vice-versa. We obtain:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$L(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} 7 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 7 \end{pmatrix}.$$

ii) $L(\mathbf{x}) = A\mathbf{x} = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$. iii) $L(\mathbf{x}) = A\mathbf{x} = \begin{pmatrix} 9 \\ 6 \\ 1 \end{pmatrix}$.

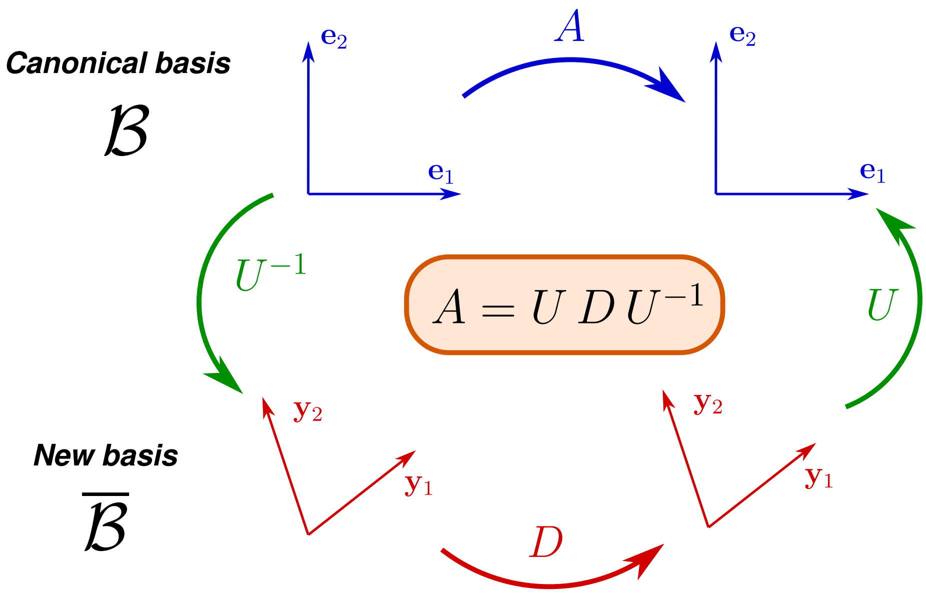

Figure 1: Formula for the change of basis.

2 Section 4.3

Ex. 1)

Denote $U = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and thus $U^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

a) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $D = U^{-1}AU = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

b) $A = D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

c) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $D = U^{-1}AU = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

d) $A = D = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

d) $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $D = U^{-1}AU = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Ex. 5)

a) $L(1) = 0$, $L(x) = x$, $L(x^2) = x2x + 2 = 2x^2 + 2$. Thus, the matrix representation of L on the basis $\{1, x, x^2\}$ is given by:

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

b) In the basis $\{1, x, x^2\}$, we obtain:

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

c) The vectors $\mathbf{u}_1 = 1$, $\mathbf{u}_2 = x$, $\mathbf{u}_3 = (1+x^2)$ of the new basis form a transition matrix:

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

using that $\mathbf{u}_3 = \mathbf{e}_1 + \mathbf{e}_3$ with $\mathbf{e}_1 = 1$, $\mathbf{e}_2 = x$ and $\mathbf{e}_3 = x^2$.

d) We use the new basis:

$$\Big(L^n(p(x))\Big)_{\overline{\mathcal{B}}} = B^n \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^n \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^2 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ a_1 \\ 2^n a_2 \end{pmatrix}.$$

In other words:

$$L^n(p(x)) = 0 \cdot \mathbf{u}_1 + a_1 \cdot \mathbf{u}_2 + 2^n a_2 \cdot \mathbf{u}_3 = a_1 x + 2^n a_2 (1 + x^2).$$

Ex. 13)

If A and B are similar, i.e. $A = PBP^{-1}$, then: $\det(A) = \det(B)$. Thus, if A nonsingular, we have $\det(A) \neq 0$ and therefore $\det(B) \neq 0$ as well. We conclude that B is nonsingular.

To show that A^{-1} and B^{-1} are similar, we perform the following computation:

$$A^{-1} = (PBP^{-1})^{-1} \quad \Rightarrow \quad A^{-1} = (P^{-1})^{-1} B^{-1} P^{-1} \quad \Rightarrow \quad A^{-1} = PB^{-1}P^{-1},$$

where we use $(AB)^{-1} = B^{-1}A^{-1}$ and $(P^{-1})^{-1} = P$. Thus, A^{-1} and B^{-1} are similar.

3 Section 5.1

Ex. 5) 2pts

Let $\mathbf{b} = (5, 2)$ and $\mathbf{x} = (1, 2)$. We try to find the closest point to $\mathbf{b}$ on the line $\mathrm{Span}(\mathbf{x})$. We use for that the projection:

$$\mathbf{p} = \frac{\langle \mathbf{b}, \mathbf{x} \rangle}{\|\mathbf{x}\|^2} \mathbf{x} = \frac{5+4}{1+4} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{9}{5} \\ \frac{18}{5} \end{pmatrix}.$$

$\boxed{\text{1+1 pts}}$

Ex. 10)

Let $\mathbf{n} = (2, 2, 1)$ the normal vector to the plane $\mathcal{P}$, i.e. the plane is given by $\text{Span}(\mathbf{n})^{\perp}$. To find the projection of the point $\mathbf{b} = (1, 1, 1)$, we first compute the distance of $\mathbf{b}$ on $\text{Span}(\mathbf{n})$ (see figure 2):

$$p = \frac{\langle b, n\rangle}{\|n\|^2} x = \frac{2+2+1}{4+4+1} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \frac{5}{9} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

The distance from b to the plane is then given by:

$$\|p\| = \frac{5}{9}\sqrt{(}2^2 + 2^2 + 1) = 5.$$

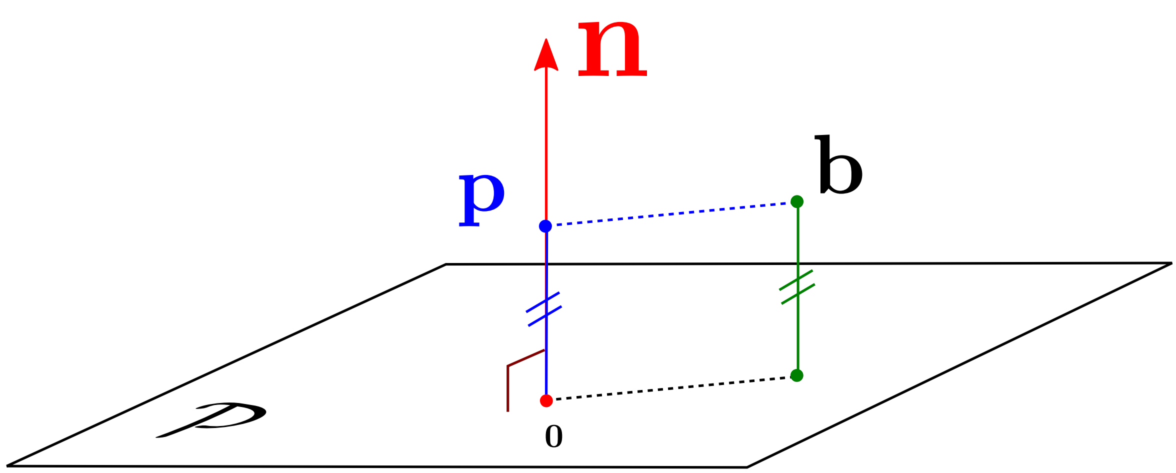

Figure 2: To find the distance of a point $\mathbf{b}$ to a plane $\mathcal{P}$, we compute its projection onto the orthogonal vector to the plane $\mathbf{n}$.