MAT 342: Homework 7 (10/18)

1 Section 5.4

Ex. 23) Take $x = (1,0)^T$.

Ex. 28) We take $a = 0$ and $b = 1$ to simplify the notation.

a) Not a norm. Take $f(x)$ continuous such that $f(0) = f(1) = 0$ and $f(.5) = 1$. The function f is not zero, but $\|f\| = |f(0)| + |f(1)| = 0$.

b) It is a norm, it is the equivalent of $\|.\|_1$ for functions.

c) It is a norm, it is the equivalent of $\|.\|_\infty$ for functions.

2 Section 5.5

Ex. 2) $\boxed{\text{3 pts}}$

a) $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = \langle \mathbf{u}_1, \mathbf{u}_3 \rangle = 0$, $\|\mathbf{u}_1\| = \|\mathbf{u}_2\| = \|\mathbf{u}_3\| = 1$.

b) Let $\mathbf{x} = (1,1,1)$. We have:

$$\begin{aligned} \mathbf{x} &= \langle \mathbf{x}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{x}, \mathbf{u}_2 \rangle \mathbf{u}_2 + \langle \mathbf{x}, \mathbf{u}_3 \rangle \mathbf{u}_3 \\ &= -\frac{2}{3\sqrt{2}} \mathbf{u}_1 + \frac{5}{3} \mathbf{u}_2 + 0. \end{aligned}$$

Thus, we deduce by the Parseval's formula that:

$$\|\mathbf{x}\| = \sqrt{\left(\frac{2}{3\sqrt{2}}\right)^2 + \left(\frac{5}{3}\right)^2} = \sqrt{\frac{2}{9} + \frac{25}{9}} = \sqrt{3}.$$

Ex. 8) $\boxed{\text{2 pts}}$

We expand the inner product:

$$\begin{aligned} \langle f, g \rangle &= \langle 3\cos x + 2\sin x \, , \, \cos x - \sin x \rangle \\ &= 3\langle \cos x \, , \, \cos x \rangle - 3\langle \cos x \, , \, \sin x \rangle + \\ &\qquad 2\langle \sin x \, , \, \cos x \rangle - 2\langle \sin x \, , \, \sin x \rangle \qquad \boxed{\text{1 pt}} \\ &= 3 - 0 + 0 - 2 \\ &= 1. \qquad \boxed{\text{1 pt}} \end{aligned}$$

Ex. 12)

Since Q is an orthogonal matrix, it preserves the inner product and the norm:

$$\langle Qx, Qy\rangle = \langle x, Q^T Qy\rangle = \langle x, y\rangle \quad , \quad \|Qx\| = \|x\|.$$

Thus, the angle between x and y is the same as the angle between Qx and Qy:

$$\text{angle}(x, y) = \text{acos}\left(\frac{\langle x, y\rangle}{\|x\|\|y\|}\right) = \text{acos}\left(\frac{\langle Qx, Qy\rangle}{\|Qx\|\|Qy\|}\right) = \text{angle}(Qx, Qy).$$

Ex. 14)

Let $H = I - 2uu^T$ (H is a reflection onto the plane $S = u^\perp$).

- H orthogonal: $H^T H = (I^T - 2(uu^T)^T)(I - 2uu^T) = I - 4uu^T + 4uu^T uu^T = I$.
- H symmetric: $H^T = I^T - 2(uu^T)^T = I - 2uu^T = H$.

Thus, $H^{-1} = H^T = H$.

Ex. 30) 3pts

a) $\langle 1, 2x - 1\rangle = \int_0^1 1 \cdot (2x - 1)\, dx = \left[x^2 - x\right]_0^1 = 0$. Thus, the functions 1 and $2x - 1$ are orthogonal for this inner product. $\boxed{1 \text{ pt}}$

b) $\|1\| = \left(\int_0^1 1 \cdot 1\, dx\right)^{\frac{1}{2}} = 1$.
$\|2x - 1\|^2 = \int_0^1 (2x - 1)^2\, dx = \left[\frac{(2x-1)^3}{6}\right]_0^1 = \frac{1}{3}$. Thus, $\|2x - 1\| = \frac{1}{\sqrt{3}}$. $\boxed{.5+.5 \text{ pt}}$

c) We use the orthonormal basis of S given by:

$$1 \quad \text{and} \quad \sqrt{3}(2x - 1).$$

Thus, the best least square approximation of $f(x) = \sqrt{x}$ onto S is given by:

$$p(x) = c_1 \cdot 1 + c_2 \cdot \sqrt{3}(2x - 1),$$

with:

$$\begin{aligned} c_1 &= \langle \sqrt{x}, 1\rangle = \left[\frac{2}{3}(x)^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}. && \boxed{.5 \text{ pt}} \\ c_2 &= \langle \sqrt{x}, \sqrt{3}(2x - 1)\rangle = \sqrt{3}\int_0^1 \sqrt{x}(2x - 1)\, dx \\ &= \sqrt{3}\left[\frac{4}{5}x^{5/2} - \frac{2}{3}x^{3/2}\right]_0^1 = \frac{2\sqrt{3}}{15}. && \boxed{.5 \text{ pt}} \end{aligned}$$

In other words, we obtain:

$$p(x) = \frac{2}{3} \cdot 1 + \frac{2\sqrt{3}}{15} \cdot \sqrt{3}(2x - 1) = \frac{4}{15} + \frac{4}{5}x.$$

3 Section 5.6

Ex. 1) $\boxed{\text{2pts}}$

a) Let $\mathbf{x}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

We have $\|\mathbf{x}_1\| = \sqrt{1+1} = \sqrt{2}$. Thus, $\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. $\boxed{\text{1 pt}}$

The projection of $\mathbf{x}_2$ on $\mathbf{u}_1$ is given by: $\boxed{\text{.5 pt}}$

$$\mathbf{p}_1 = \langle \mathbf{x}_2, \mathbf{u}_1 \rangle \mathbf{u}_1 = \frac{-3+5}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Thus, $\mathbf{x}_2 - \mathbf{p}_1 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and $\|\mathbf{x}_2 - \mathbf{p}_1\| = 4\sqrt{2}$. We can take for $\mathbf{u}_2$ the vector: $\boxed{\text{.5 pt}}$

$$\mathbf{u}_2 = \frac{\mathbf{x}_2 - \mathbf{p}_1}{\|\mathbf{x}_2 - \mathbf{p}_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

b) Let $\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$.

Using a similar procedure, we find:

$$\mathbf{u}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{u}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$